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DISSERTATION

SATELLITE MOTION AROUND AN OBLATE PLANET: A PERTURBATION SOLUTION FOR ALL ORBITAL PARAMETERS

by

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To Whom It May Concern:

After Dr Snider's graduation, I discovered some algebraic errors in his solution. We have eliminated the problem in the equatorial and polar orbit cases and will publish these special cases in the Proceedings of the AAS/AIAA Astrodynamics Conference held in Stowe, Vermont, on August 7-10, 1989. We are currently working on the general case and will soon publish a paper with the correct results for orbits of arbitrary inclination.

The method of solution outlined herein is correct and an original contribution to the literature, but the actual results herein should not be taken literally.

D. A. Danielson

DONALD DANIELSON
Associate Professor of Mathematics
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The search for a universal solution of the equations of motion for a satellite orbiting an oblate planet is a subject that has merited great interest because of its theoretical implications and practical applications. The discovery of such a solution should motivate a reassessment of both the theories that exhibit singularities and the physical effects implied by singularities. The practical importance of such a solution is the efficiency of simple analytic formulas in predicting simultaneously the paths of large numbers of satellites in a multitude of orbits. Here, a complete first order solution to the problem of a satellite, perturbed only by the oblateness of the Earth, is displayed. The orbit is free of singularities for all parameters and is valid for 1000 revolutions with a relative error of the order $J^2 \approx 10^{-6}$.

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Satellite Motion Around an Oblate Planet: A Perturbation Solution for All Orbital Parameters

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ABSTRACT

The search for a universal solution of the equations of motion for a satellite orbiting an oblate planet is a subject that has merited great interest because of its theoretical implications and practical applications. The discovery of such a solution should motivate a reassessment of both the theories that exhibit singularities and the physical effects implied by singularities. The practical importance of such a solution is the efficiency of simple analytic formulas in predicting simultaneously the paths of large numbers of satellites in a multitude of orbits. Here, a complete first order solution to the problem of a satellite, perturbed only by the oblateness of the Earth, is displayed. The orbit is free of singularities for all parameters and is valid for 1000 revolutions with a relative error of the order $J^2 \approx 10^{-6}$.

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NOTATION

a semi major axis of the initial instantaneous ellipse

c $\cos i_0$

e eccentricity of the initial instantaneous ellipse ($0 \leq e \leq < 1$)

$\bar{h}$ constant value of $r^2 \frac{\cos^2 \delta}{\cos i_0} \frac{d\phi}{dt}$ along an orbit, approximately equal to the initial magnitute of angular momentum

i inclination of reference plane

i_0 initial value of i

J_2 oblateness coefficient of the planet (coefficient of the second harmonic in the expansion of the gravitational potential)

J $\frac{3}{2}\left(\frac{J_2 R^2}{\bar{p}^2}\right)$

$\bar{p}$ $\bar{h}^2/GM$ where G is the gravitational constant and M is the mass of the planet

R equatorial radius of the planet

r radial distance from the center of the planet to the satellite

s $\sin i_0$

t time

t_0 initial value of t

u $\frac{\bar{p}}{r}$

V gravitational potential

δ latitude of the satellite

ϕ longitude of the satellite

θ angle from line NON' to the satellite measured in the reference plane (Fig. 5.1)

θ_0 initial value of θ

Ω longitude of line NON' (Fig. 5.1)

Ω_0 initial value of Ω

ω argument of perigee of the initial instantaneous ellipse

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I. INTRODUCTION

A characteristic feature of practical satellite orbit prediction is that the engineer may deal with numerous satellites in a great variety of alternative orbits. Under these and many other such circumstances analytic relations which can quickly approximate an orbit may be far superior to large numerical models. While many analytic methods have been developed for the artificial satellite age, most are not used in practical orbit prediction because they violate one or more of the following principles:

- The method should provide a solution that is significantly more accurate than the two-body solution.
- The real physical effects of the orbit should be easily distinguishable in the solution.
- The solution should be universal; it should be valid for all orbital parameters.

The motivation for this research was the desire to develop a method for satellite prediction that would embody these characteristics.

In this analysis, a solution to the equations of motion of a satellite around an oblate planet is found by use of a variation of the perturbation technique known as the Method of Strained Coordinates. The orbit is valid for 1000 revolutions with a relative error of 10^{-6}. The solution which is valid for all eccentricities and for all inclinations, was obtained by extensive use of the symbolic manipulation program MACSYMA.

The analysis begins with a background discussion of some of the competing satellite orbit theories. There is then a development of the equations of motion beginning with a derivation of the two-body solution. The various forces which act

to disturb the two-body orbit are highlighted; a more thorough discussion is given for the effects of oblateness. There is a complete treatment of the perturbation technique as the equations of motion are solved in detail. The complete first order solution is displayed as a function of coordinates and as a function of the orbital elements. In addition, an independent analysis of the polar and equatorial orbits is performed to serve as a check of the general solution.

II. BACKGROUND

A. INTRODUCTION

The theory of flight of artificial satellites is closely related to classical celestial mechanics, one of the oldest and most highly developed branches of science. The equations which describe the motion of an artificial satellite are in principle identical to the equations of motion of natural celestial bodies. It is not surprising then that the results originally derived in classical celestial mechanics have been freely used to explain the motion of artificial celestial bodies.

The foundations of classical celestial mechanics were established in the eighteenth century when Clairaut, d'Alembert, Laplace, Lagrange, and Euler introduced theories and analytical methods to explain the large deviation of the Moon from an elliptic orbit due to solar attraction. These theories all supplemented or complemented the pioneer work that had been done by Newton. Newton had correctly indicated the Moon's variation in eccentricity and inclination and the regression of the nodes; however, his published theory accounted for only half of the motion of perigee. (A century later an unpublished work was found to contain the full explanation.) Clairaut in 1749 was on the verge of substituting a new law of gravitation for the Newtonian law when he found that second order perturbations removed the discrepancy in the motion of perigee.

In that same century, Euler investigated and developed the perturbative function and began the development of the method of variation of parameters, which was later extended by Lagrange.

Lagrange, Laplace, and Poisson all advanced the discipline through their investigation of the stability of the solar system. Later, in the nineteenth century, the use of Hamiltonian mechanics was used to great advantage by Delaunay whose work influenced the innovators of artificial satellite orbit prediction, namely Brouwer, Kozai, and Garfinkel.

The introduction of the modern computer and the launch of the first artificial satellites had a profound effect on the science of celestial mechanics. Classical celestial mechanics had been essentially a contemplative science with the principle aim being to study the laws of motion of existing heavenly bodies. In contrast, the science of the flight of artificial satellites is an active engineering science concerned with determining or predicting relatively short term orbits, and in many cases controlling the satellite's motion through on-board propulsive devices. It was in the exciting climate, following the successful launchings of the first artificial satellites, that most of the new methods of satellite orbit prediction were developed.

B. THE USE OF HAMILTONIAN MECHANICS IN SATELLITE ORBIT PREDICTION

There are certain schools of thought in dynamic astronomy and theoretical physics that support the loyal use of Hamiltonian mechanics [Ref. 1, p. 228], and many of the new methods in general perturbations take advantage of the elegant formalism offered by the Hamiltonian method. The Hamiltonian method is referred to here as the formal process of writing the equations of motion for a satellite in the canonical form:

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r} \qquad \frac{dp_r}{dt} = \frac{-\partial H}{\partial q_r} \qquad (r = 1, 2 \ldots 3n) \tag{2.1}$$

where $q_r \equiv$ generalized coordinates

$$p_r \equiv \text{generalized momenta}$$
$$H = \sum_1^{3n} p_r \frac{dq_r}{dt} - L$$

(H is the Hamiltonian and L is the Lagrangian, $L = T - V$, kinetic energy - potential energy).

The solution to (2.1) may be written down if a function S can be found, where S is any complete solution of the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(q_r, \frac{\partial S}{\partial q_r}, t\right) = 0 \tag{2.2}$$

It should be noted that (2.2) is tractable only if the variables q_r and t are separable within S and H.

Although this present analysis does not use Hamiltonian mechanics to solve the equations of motion, a summary of its use is merited since the advances in this area have been an invaluable contribution to general perturbation theory.

In celestial mechanics, one may formulate a Hamiltonian that represents the gravitational attraction of the central force and add to it terms which are "perturbing Hamiltonians." Various sets of canonic variables may be chosen with the goal of expressing a zero-order Hamiltonian in a simple form and the higher order effects in an iterative fashion. Each term then may be dealt with through a succession of canonic transformations. Delauney introduced a systematic procedure for isolating parts of the Hamiltonian and then generating a suitable transformation in successive steps. A particular feature of his approach is that a periodic term in the Hamiltonian may be eliminated with each canonical transformation [Ref. 2]. While DeLauney used a procedure that eliminated one term at a time, von Ziepel devised a technique that eliminates one angular variable with each transformation. This method reduces the number of degrees of freedom while at the same time

imparting to the transformed Hamiltonian a symmetry of the unperturbed system [Ref. 1]. The eliminated variables are referred to as ignorable coordinates since they do not participate in the solution of the transformed equations of motion but can be recovered after the solution has been obtained.

Using von Ziepel's method, Brouwer devised one of the most notable general perturbation theories [Ref. 3]. Prior to Brouwer's method, all previous work in artificial satellite theory had written the Hamiltonian as a Fourier series in the mean anomaly with coefficients that were infinite series in powers of the eccentricity. Brouwer used an elliptic approximation for the potential and obtained a complete first order theory with some second order development using canonical transformations. The essence of Brouwer's method was to write equations (2.1) as

$$\frac{dL_j}{dt} = \frac{\partial H}{\partial l_j} \qquad \frac{dl_j}{dt} = -\frac{\partial H}{\partial L_j} \tag{2.3}$$

where

$$\begin{array}{ll} L_1 = \sqrt{GMa} & l_1 = M \\ L_2 = L_1\sqrt{1-e^2} & l_2 = \omega \\ L_3 = L_2 \cos i & l_3 = \Omega \end{array}$$

are the Delauney variables where

a - semimajor axis	e - eccentricity
i - inclination	M - mean anomaly
ω - argument of perigee	Ω - longitude of ascending node

Referring back to the Hamilton Jacobi equation (2.2), the function S is used as a generating function to find a new Hamiltonian that leads to simplified canonic equations. By choosing S correctly, Brouwer was able to find a canonical transformation from the Delauney variables to a set of double primed Delauney variables (L_j'', l_j'') such that (2.3) have the form

$$\frac{dL_j''}{dt} = 0 \qquad \frac{dl_1''}{dt} = -\frac{\partial H^*}{\partial L_j} \tag{2.4}$$

where H^* is the Hamiltonian expressed in terms of double primed orbital elements by

$$L_1'' = \sqrt{GMa''} \qquad l_1'' = M''$$

$$L_2'' = L_1''\sqrt{1 - e''^2} \quad l_2'' = w''$$

$$L_3'' = L_2'' \cos i'' \qquad l_3'' = \Omega''$$

The double primed orbital elements are related to the unprimed elements by

$$a = a'' + \epsilon a \qquad e = e'' + \epsilon e$$

$$i = i'' + \epsilon i \qquad M = M'' + \epsilon M$$

$$w = w'' + \epsilon w \quad \Omega = \Omega'' + \epsilon \Omega$$

where the quantities $\epsilon a, \epsilon e, \epsilon i, \epsilon M, \epsilon w, \epsilon \Omega$, are periodic functions of the double primed orbital elements.

Equations (2.4) are solved for the double-primed Delauney variables which can then be expressed in terms of the original variables.

The results initially obtained by Brouwer were not valid at the critical inclination of 63°.4, and they were questionable when either the inclination or eccentricity were near zero [Ref. 4]. O. K. Smith devised a method for dealing with the problem of zero inclination and eccentricity [Ref. 5], but Brouwer subsequently challenged the validity of his method. Later, Lyddane [Ref. 6] was able to successfully remove the restrictions on small values of eccentricity and inclination by reformulating Brouwer's work in terms of an alternate set of variables.

Brouwer used the central force term as the first approximation for the potential since there is no exact solution to the equations of motion of a satellite

under the influence of the more complex potential described by an oblate planet having axial symmetry. Other authors have attempted to introduce a new potential which approximates the Earth's potential better than the central force term alone and also leads to an exact solution. Most notable has been the work of Sterne [Ref. 7] and Garfinkel [Ref. 8]. Sterne's potential function accounts for most of the standard potential through the second harmonic and it leads to a solution of canonical constants that are free of first order secular perturbations. The remaining effects of the earth's oblateness and other forces are allowed for in the perturbing Hamiltonian which causes the six canonical constants of the unperturbed solution to undergo variations with time. Sterne provided the inspiration for Garfinkel's method which is essentially the same as Sterne's but more developed. Garfinkel included the second and fourth harmonics and arrived at a solution that is reducible to quadratures. Garfinkel's original solution was not valid at the critical inclination, but in a later paper he removed the singularity through a variation on his perturbation theory [Ref. 9].

While Garfinkel did his analysis in spherical coordinates, Vinti [Ref. 10], derived a potential expressible in oblate spheroidal coordinates. In his original analysis, Vinti introduced a potential function and associated coordinate system that would lead to the separability of the Hamilton-Jacobi equations. In a subsequent work [Ref. 11], Vinti showed that the equations result in a solution reducible to quadratures. The Vinti potential is an exact expression of the Earth's potential through the second harmonic. The theory provides perturbed coordinates, not perturbed elements. In his second order analysis [Ref. 12], Kozai criticized this characteristic, and he chose not to use the Vinti potential since it would require changing the definitions of the conventional orbital elements.

Morrison [Ref. 13] showed that the von Zeipel method is a particular case of the method of averaging, and Liu [Ref. 14] used the latter technique to study the combined effects of air drag and planet oblateness on a satellite orbit. The method of averaging, unlike von Ziepel's method, does not require that transformations be canoı '-al. The method of averaging has been used extensively in recent years by Lorell and Liu [Ref. 15], McClain [Ref. 16], and Hoots [Ref. 17]. However, the validity of the method has been challenged; most notably Taff [Ref. 18] doubts its rigorous foundation. Arnold [Ref. 19] notes that the principle of averaging is a vaguely formulated and rigorously untrue assertion, but he adds that sometimes such assertions are fruitful mathematical sources. It should be also noted that the solutions obtained in Liu's analysis [Ref. 14] are not valid for near circular equatorial orbits nor at the critical inclination, while those obtained by Hoots are valid only for small eccentricities ($0 < e < .1$), and they are invalid at inclinations near 0° or the critical.

Hamiltonian mechanics has provided a rich source of literature in orbit theory; however its practical applicability has been questioned in some textbooks. Roy [Ref. 20] briefly outlines the use of the canonic equations, while Taff [Ref. 18p. 322] states that he does not see any additional practical applications provided. Baker [Ref. 21] chooses not to represent the subject. A general criticism is that the process of generating suitable transformations in the perturbation procedure tends to make the coordinates and the momenta less distinguishable on physical grounds and more difficult to relate to the set of natural coordinates which were used to write down the initial set of differential equations. While the ultimate form of the governing equations may be simple to solve, there remains the tedious task of obtaining explicit results in terms of physically meaningful coordinates or elements.

C. KOZAI'S METHOD

Prior to his work cited above, Kozai [Ref. 22] developed a method for finding the perturbations on the orbital elements of a satellite considering only the oblateness of the Earth. Kozai developed a disturbing function based on the Earth's departure from a sphere, and he used a version of Lagrange's planetary equations to formulate the solution. Kozai used the standard form for the Earth's potential and included the harmonics J_2, J_3, and J_4. Despite the use of the higher harmonics, the theory is first order. Kozai expressed short-period terms in J_2, the secular in J_2, J_4, and J_2^2, and the long period terms in $J_2, \frac{J_3}{J_2}$, and $\frac{J_4}{J_2}$. The analytic expressions are developed using the standard orbital elements.

Kozai's work is cited here because, due to its simplicity, the method has become very popular in many textbooks and handbooks on orbital mechanics. [Refs. 20, 23, 24]. However, Taff cites Kozai's method as an example of misapplication of perturbation theory [Ref. 18p. 332]. Taff challenges the assumptions made by Kozai in his analysis, and he points out that the method is invalid at the critical inclination.

As was stated in Chapter I, a motivation for this current analysis was the purpose of finding a perturbation method that would lead to universal solution. All of the methods discussed thus far have particular problems at certain inclinations or eccentricities. Some of the problem cases have been resolved by unique efforts (These cases are discussed in Appendix C.); however, one should question the underlying validity of any perturbation method that produces singularities. There has been no satisfactory way found for avoiding the critical inclination singularity in Kozai's method.

D. THE DIRECT METHODS

R. E. Roberson [Ref. 25] devised an approach for finding the qualitative and approximate quantitative results concerning the behavior of a set of orbital elements in the gravitational field of an oblate planet. The motion of a near satellite around the planet is simply described to first order by introducing a frame of reference which contains a mean orbital plane having a constant inclination. Both the reference frame and the orbit plane rotate at a constant angular velocity. With respect to this doubly moving reference frame, the motion of the satellite differs from pure elliptic motions by only periodic perturbations.

King-Hele [Ref. 26] advanced the approach taken by Roberson. He introduced a non-rotating reference frame with a orbit plane that continually rotates at a non-constant rate about the Earth's axis. A relation is found between the rotating orbit plane and the angular rate of travel of the satellite. The equations of motion are written in position coordinates, but are subsequently rearranged in terms of a modified set of orbital elements. The inclination of the rotating reference plane is held strictly constant in the analysis. King-Hele formulated the problem by employing a power series expansion in terms of the eccentricity; therefore, the method is limited to small eccentricities. The final solution contains an incomplete set of workable elements and hence the method gives mostly a qualitative description of the satellite's behavior due to oblateness.

King-Hele's analysis was the inspiration for the work of Brenner and Latta [Ref. 27]. They improved King-Hele's original analysis by abandoning the condition of constant orbit inclination and by retaining the eccentricity in closed form expressions. Brenner and Latta limited their analysis to small eccentricity although the method is not so restricted. They obtained an approximate first order solution and demonstrated that the method is valid for higher order analysis.

An advantage of the direct method is that one may use ordinary perturbation analysis (the Method of Strained Coordinates) to solve the equations of motion. In addition the chosen set of orbital elements used throughout the analysis correspond closely with the classical elements; therefore, physical interpretation is facilitated.

The disadvantage in using the method is shared by most all other procedures that must use a perturbation scheme. The process requires the manipulation of massive algebraic expressions. However, this drawback has been greatly reduced by the introduction recently of large symbolic mathematics programs, such as MACSYMA, that handle the bookkeeping.

III. THE TWO-BODY PROBLEM

A. INTRODUCTION

The central problem of celestial mechanics is the two-body (or Kepler) problem. Simply stated, the problem is to solve for the motion of two particles interacting through their mutual gravitation in an isolated space. A solution of the two body problem often represents physical reality in an acceptable way. For instance, the orbit of the Earth around the Sun may be treated, to a first approximation, as a two-body problem because the influence of perturbing bodies, the Moon, Jupiter, etc., are small compared with the Sun's gravitational attraction. Likewise, more complex problems such as a spacecraft mission to Mars, a four body problem – Earth, Sun, Mars, spacecraft, may be treated by breaking the flight into three two-body problems. Such a technique is used in the patched conic method where two-body solutions are literally patched together.

There are other more important reasons for studying the two-body problem. It is the only gravitational problem in dynamics, other than very specialized cases in the three-body problem, for which there is a complete and general solution, and it is possible to gain considerable insight into more general phenomena of motion by a thorough study of the two-body problem. In fact, the most complete theories of celestial motion use functions appearing in the solution of the two-body (elliptic case) problem as elementary functions. The solution is central to this present analysis because it will serve as a starting point for the generating of analytical solutions that are valid to higher orders of accuracy. These solutions, called general perturbation theories, are the subject of Chaper VI.

In this chapter, the equations for the two-body problem will be derived and solved. The first step is to chose a coordinate system in which the laws of Newton hold (an inertial coordinate system). In practice, the reference frame of the "fixed" stars provides a very good approximation to an inertial reference frame. Next, following the method of Nelson and Loft [Ref.28 pp.82–84] it will be shown that the center of mass of two bodies in this coordinate system is also an inertial reference point. Then the equations of motion will be derived and solved using the polar angle θ as the independent variable.

B. THE DIFFERENTIAL EQUATION

Figure 3.1 shows two mass centers at position $\mathbf{r}_1$ and $\mathbf{r}_2$. The origin O is defined to be an inertial reference point. The distance between the two mass centers is $\mathbf{r}$ where

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1.$$

Combining Newton's third law of motion and his law of universal gravitation gives

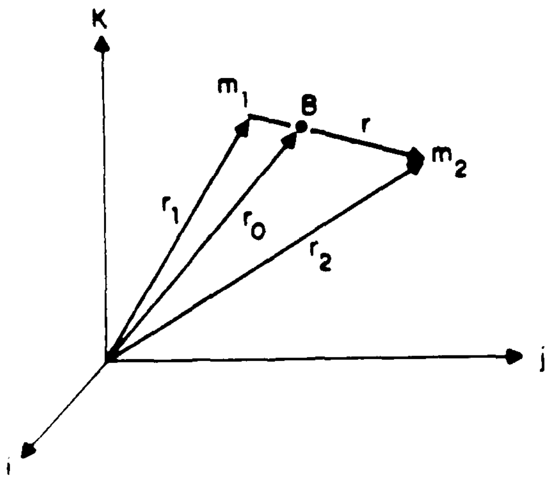

Figure 3.1: Two Body Problem

the force equations for the two bodies:

$$m_1 \frac{d^2\mathbf{r}_1}{dt^2} = \frac{Gm_1 m_2 \mathbf{r}}{r^3} \tag{3.1}$$

$$m_2 \frac{d^2\mathbf{r}_2}{dt^2} = \frac{-Gm_1 m_2 \mathbf{r}}{r^3} \tag{3.2}$$

where G is the universal gravitational constant.

Adding (3.1) and (3.2) and integrating results in

$$m_1 \frac{d\mathbf{r}_1}{dt} + m_2 \frac{d\mathbf{r}_2}{dt} = \text{constant.} \tag{3.3}$$

Now the center of mass of the system (barycenter) in Fig. 3.1 is defined as:

$$(m_1 + m_2)\mathbf{r}_0 = m_1\mathbf{r}_1 + m_2\mathbf{r}_2. \tag{3.4}$$

By combining equations (3.3) and (3.4) the following result is obtained,

$$\frac{d\mathbf{r}_0}{dt} = \frac{m_1}{m_1 + m_2}\frac{d\mathbf{r}_1}{dt} + \frac{m_2}{m_1 + m_2}\frac{d\mathbf{r}_2}{dt} = \text{constant.}$$

So the barycenter moves with constant velocity, and it too is an inertial reference point. Subtracting (3.1) from (3.2) results in

$$\frac{d^2\mathbf{r}_2}{dt^2} - \frac{d^2\mathbf{r}_1}{dt^2} = \frac{d^2\mathbf{r}}{dt^2} = \frac{-G(m_1 + m_2)\,\mathbf{r}}{r^3}, \tag{3.5}$$

the solution of which gives the position of either body relative to the other. Now choose the barycenter as the origin and define the position of m_1 and m_2, respectively, as

$$\mathbf{r}_{01} = \mathbf{r}_1 - \mathbf{r}_0 = \frac{-m_2}{m_1 + m_2}\mathbf{r}$$

$$\mathbf{r}_{02} = \mathbf{r}_2 - \mathbf{r}_0 = \frac{m_1}{m_1 + m_2}\mathbf{r}.$$

Substitution of the above equations in (3.1) and (3.2) yield separate equations for the motion of each body relative to the barycenter:

$$
\begin{aligned}
\frac{d^2\mathbf{r}_0}{dt^2} + \frac{d^2\mathbf{r}_{01}}{dt^2} = \frac{d^2\mathbf{r}_{01}}{dt^2} &= \frac{-G\,m_2^3}{(m_1+m_2)^2 r_{01}^3}\,\mathbf{r}_{01} \\
\frac{d^2\mathbf{r}_0}{dt^2} \quad \frac{d^2\mathbf{r}_{02}}{dt^2} = \frac{d^2\mathbf{r}_{02}}{dt^2} &= \frac{-G\,m_1^3}{(m_1+m_2)^2\, r_{02}^3}\,\mathbf{r}_{02}.
\end{aligned}
\tag{3.6}
$$

Equation (3.5) and equations (3.6) are of identical form, differing only by a constant, so that

$$\frac{d^2\mathbf{r}}{dt^2} + \frac{GM\,\mathbf{r}}{r^3} = 0 \tag{3.7}$$

is the vector differential equation of motion for either of the two bodies. $\mathbf{r}$ is the distance to the other body, or to the barycenter, according to the appropriate choice of GM. For the problem of a satellite orbiting the earth, the mass of the satellite can be neglected in comparison with the mass of the earth, therefore GM is the product of the universal constant and the mass of the earth.

C. THE INTEGRATION OF THE TWO-BODY PROBLEM

There are no cross products involved in equation (3.7); therefore, all motion must lie in the plane that contains $\mathbf{r}$ and $\frac{d\mathbf{r}}{dt}$. The scalar components of acceleration are

$$r\frac{d^2\theta}{dt^2} \quad + \quad \frac{2dr}{dt}\frac{d\theta}{dt} = 0 \tag{3.8}$$

$$\frac{d^2r}{dt^2} \quad - \quad r\left(\frac{d\theta}{dt}\right)^2 = \frac{-GM}{r^2}. \tag{3.9}$$

Writing (3.8) as $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$ and integrating yields

$$r^2\frac{d\theta}{dt} = h = \text{ constant} \tag{3.10}$$

where h is the specific angular momentum.

Equation (3.10) is an exact integral of (3.8). It corresponds to Kepler's empirical law of constant areal velocity which states that the area swept out by the radius vector of a planet is uniform in time.

From (3.10), the independent variable may be changed from t to θ, e.g.,

$$\frac{d}{dt} = \frac{h}{r^2}\frac{d}{d\theta}$$

so (3.9) becomes

$$-\frac{1}{r^2}\frac{d^2r}{d\theta^2} + \frac{2}{r^3}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r} = \frac{GM}{h^2} = \frac{d^2}{d\theta^2}(1/r) + \frac{1}{r} = \frac{GM}{h^2}.$$

Since this equation is linear in the reciprocal of r, it may be written as

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$$

which has the solution

$$u = \frac{GM}{h^2} + A\cos(\theta - w) \tag{3.11}$$

or reintroducing r, (3.11) becomes

$$r = \frac{h^2/GM}{1 + (Ah^2/GM)\cos(\theta - w)}$$

which may be written as the polar equation of a conic section

$$r = \frac{p}{1 + e\cos(\theta - \omega)} \tag{3.12}$$

so that

$$p = h^2/GM \text{ and } e = Ah^2/GM.$$

D. ELLIPTIC MOTION

Equation (3.12) is the equation of a conic with prime focus at O. The conic has a semi-latus rectum h^2/GM, an eccentricity e, and a semi-major axis a that makes an angle $f = \theta - \omega$ with the horizontal axis (Figure 3.2). The extreme endpoints of the major axis of the orbit are referred to as apsides or apses. The point nearest the prime focus is called perigee and is given by $\theta = \omega$. The point farthest from the prime focus is given by $\theta = \omega + 180°$ and is called apogee. The angle ω, "argument of perigee", will be discussed later in this chapter.

The energy of the satellite in the orbit is conserved and is equal to

$$E = m_s(v^2/2 - GM/r) = m_sC$$

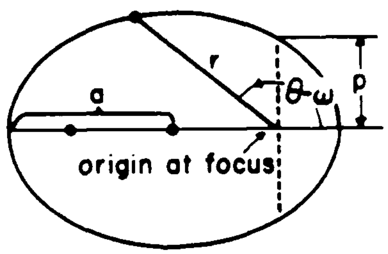

Figure 3.2: The Elliptic Orbit

where C is the total energy of the satellite, $v^2/2$ the kinetic energy and $-GM/r$ the potential energy of the satellite, all per unit mass (m_s). In general C is equal to $-GM/2a$, so that the satellite's energy depends only on the semi-major axis. From this relationship it is easily shown that the satellite's velocity is given by

$$v^2 = GM(2/r - 1/a).$$

Then the relationships $r_p = a(1-e)$ and $r_a = a(1+e)$ result in

$$v_p^2 = \frac{GM(1+e)}{a(1-e)}$$

and

$$v_a^2 = \frac{GM(1-e)}{a(1+e)}$$

so that the velocity is a maximum at perigee and a minimum at apogee.

The area of the ellipse is $\pi a^2 \sqrt{(1-e^2)}$, and the rate of description of area is $\frac{r^2}{2}\frac{d\theta}{dt} = \frac{h}{2}$. Since $h^2 = GMa(1-e^2)$, the orbital period may be written as

$$T = \frac{2\pi}{\sqrt{GM}} a^{3/2}.$$

By defining the mean motion n as $T = \frac{2\pi}{n}$, so that $n^2 a^3 = GM$, one may proceed to derive an expression for position versus time in the elliptic orbit.

The orbital ellipse APB with center at C touches at perigee, A, and at apogee, B, a concentric circle also centered at C which has as a radius the semi-major axis of the ellipse. The circle C is known as the auxiliary circle, and is geometrically related to the ellipse by the relation

$$PN = P'N\sqrt{(1-e^2)}$$

where e is the eccentricity, P any point on the ellipse, N the foot of the perpendicular through P upon AB, and P' the intersection of this perpendicular with the circle C. The angle $ACP' = E$ is known as the eccentric anomaly (Figure 3.3).

Let τ be the time of perigee passage and t the time, then $t - \tau$ is the time since perigee passage. The quantity,

$$M = n(t - \tau), \tag{3.13}$$

is called the mean anomaly. Using the geometry of Figure 3.3, one may derive the following relationship between the anomalies:

$$M = E - e \sin E, \tag{3.14}$$

which is known as Kepler's equation. If the position of a satellite in a fixed orbit relative to the earth is desired at some specified time t, then equation (3.13) gives M and (3.14) gives E. The distance to the satellite is found by the relationship

$$r = a(1 - e \cos E).$$

The angular position of the satellite is defined by the true anomaly, $\theta - w = f$, or the angular distance from perigee:

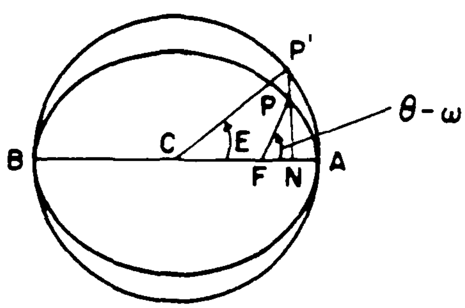

Figure 3.3: The Eccentric Anomaly

$$\tan(f/2) = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\frac{E}{2}.$$

In actual practice, the second step in the above process, solving (3.14) is a bit more involved since a closed form solution to Kepler's equation does not exist. However, dozens of methods of successive approximations have been devised. For instance, by use of a numerical method M can be calculated for a few values of E and then the correct E can be found by inverse interpolation.

E. CONSTANTS OF THE MOTION

In section B, the original equations of motion (3.1) and (3.2) were reduced to (3.7). Thus the problem was reduced from one of three second order differential equations requiring twelve constants to one of three second order equations with six constants. A discussion of the constants of motion, some of which were introduced during the solution of the two-body equation, is the purpose of this section. The six constants may be written in a variety of forms, a choice among forms is usually made with the purpose of simplifying the problem.

Equation (3.7) was solved by the classic technique of changing the independent variable from t to θ. But (3.7) in its present form is integrable; that is, there exists sufficient time independent first integrals, or functions that are constant along the motion, to specify each orbit. The first of these is the angular momentum. Cross-multiplying (3.7) by $\mathbf{r}$ results in

$$\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} + \mathbf{r} \times \frac{GM}{r^3}\mathbf{r} = 0$$

or

$$\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = 0.$$

And since

$$\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=\frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}+\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$$

then

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{v})=0.$$

By integration

$$\mathbf{r}\times\mathbf{v}=\mathbf{h}$$

where $\mathbf{h}$, the angular momentum, provides three constants of motion. Similarly, by cross-multiplying (3.7) by $\mathbf{h}$, one may obtain the Lenz vector

$$\mathbf{e}=\frac{d\mathbf{r}}{dt}\times\frac{\mathbf{h}}{GM}-\frac{\mathbf{r}}{r}\qquad\left(\frac{d\mathbf{e}}{dt}=0\right)$$

where $\mathbf{e}$ is a vector along the major axis of the orbit pointing toward the position of closest approach or perigee ($|\mathbf{e}|=e$). The vector $\mathbf{e}$ provides only two independent constants since $\mathbf{h}$ and $\mathbf{e}$ are perpendicular vectors ($\mathbf{h}\cdot\mathbf{e}=0$), and one remaining constant is required.

The vector integrals $\mathbf{h}$ and $\mathbf{e}$ specify that the orbit will lie in the plane perpendicular to $\mathbf{h}$ and have a shape determined by $\mathbf{e}$. The classical orbital elements: a (semi-major axis), e (eccentricity), i (inclination), Ω (longitude of the ascending node), and ω (argument of perigee), may be derived from these vectors.

From (3.12) the equation for r may be written as

$$r=\frac{h^2/GM}{1+e\cos f}$$

where f is the angle between $\mathbf{e}$ and $\mathbf{r}$ (the true anomaly).

Restricting e to the elliptic case results in

$$r_1 = \frac{h^2}{GM(1+e)} \le r \le \frac{h^2}{GM(1-e)} = r_2$$

so that the major axis of the ellipse is

$$2a = r_1 + r_2 = \frac{2h^2}{GM(1-e^2)}.$$

Knowing e and a gives the shape and size of the orbit. The orbit now can be oriented in a coordinate system. Reference is made to Figure 3.4 and the two angles i and Ω. i is the inclination of the orbit plane defined as the angle between the equatorial plane and the orbit plane. Since this is the same angle as the angle between the z axis ($\mathbf{k}$ unit vector) and the angular momentum vector $\mathbf{h}$, i may be found by

$$\cos i = \frac{h_k}{h}.$$

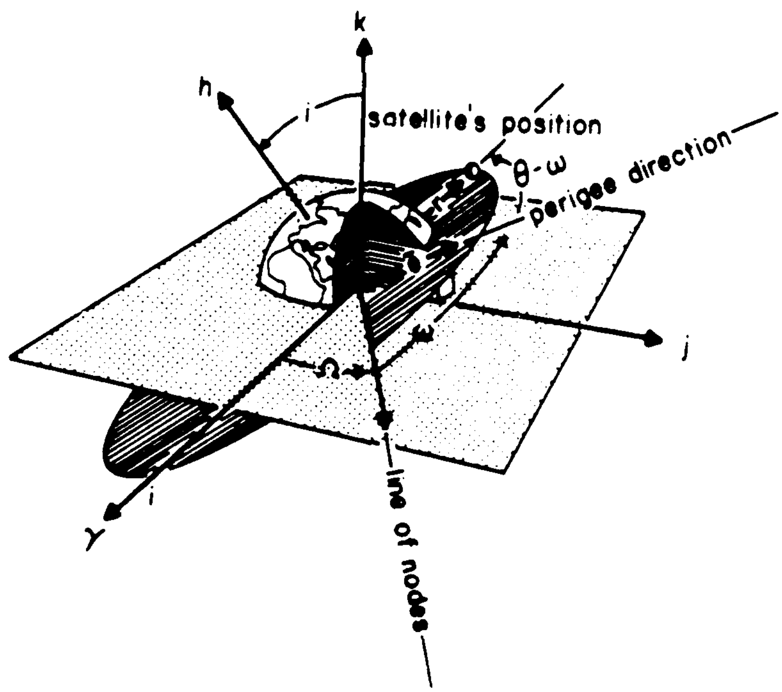

Figure 3.4: Constants of the Orbit

The angle Ω, the longitude of the ascending node, is the angle in the equatorial plane, between the $\mathbf{i}$ unit vector and the point N where the satellite crosses through the equatorial plane in a northerly direction measured counterclockwise when viewed from the north side of the equatorial plane. The vector $\mathbf{n}$ lies along NO such that

$$\mathbf{n} = \mathbf{k} \times \mathbf{h}.$$

Therefore Ω may be found by

$$\cos \Omega = \frac{n_i}{n}.$$

As was stated above, the vector $\mathbf{e}$, points toward perigee. The angle ω (the argument of perigee) measures the distance between NO and perigee and can be found by

$$\cos \omega = \frac{\mathbf{n} \cdot \mathbf{e}}{ne}.$$

With the constants a, e, i, Ω, ω specified, the orbit is defined in the coordinate system. The remaining task, to locate the satellite in the orbit at any time, requires one more constant.

The final constant of motion is given by the relationship between the magnitude of the angular momentum and the true anomaly (f):

$$\int_{\tau}^{t} h dt = \int_{0}^{f} r^2 df.$$

By a change of variables from f to the eccentric anomaly E, the above equation may be easily integrated. The result by this analytical method is identical to equation (3.14), Kepler's equation, which was previously derived by geometry. The constant τ, time of perigee passage, is the final constant of integration.

F. SUMMARY

This completes the analysis of the two-body problem. It has been shown that a combination of six constants will strictly define the motion of a satellite under the influence of a central gravitational force. The six that were chosen, referred to now as the orbital elements, can be used to find other constants of the motion including **r** and **v**, the distance and velocity vectors of the satellite.

IV. SATELLITE PERTURBATIONS

A. INTRODUCTION

As was demonstrated in the last chapter, the classical two-body problem has solutions that can be written in closed form when the polar angle (or the eccentric or true anomaly) is used as the independent variable. If an additional force acting on either of the two bodies is introduced, the resulting equations of motion usually no longer have closed-form solutions. When the magnitude of such a force is small compared to the central gravity term, the force is called a perturbation. The resulting orbit experiences a small departure from the Keplerian orbit, at least initially. These departures are also called perturbations. Under certain circumstances, it is possible to make analytic approximations to the effects of the perturbing forces, though a precise solution cannot be obtained. Generally, the methods consist in determining the exact equations of motion and then assuming that their solutions do not depart appreciably from the case of no disturbing force. Then only an indication of the actual motion of the body can be obtained. Precise solutions can be found for specific initial conditions by numerical integration techniques, but these solutions give little insight into the dependence of the motion on the parameters of the disturbing force. In some cases, the approximations obtained with analytic methods may exceed the precision of numerical methods, especially if the prediction is required for a long period of time and there is a clear dominance of one particular force.

In the case of a close satellite about a non-spherical planet, a potential function V can be formed such that

$$V = V_0 + R$$

where V_0 is the potential function due to the two-body problem and R the disturbing function that is at least an order of magnitude less than V_0. Many general perturbation theories make use of the fact that the two-body orbit of the body due to V_0 changes slowly due to R, and they attempt to obtain analytical expressions for the changes in the orbital elements due to R within a specified time interval. If the elements of an elliptical orbit are $a_0, e_0, i_0, \Omega_0, w_0$, and t_0, the ellipse with these elements is referred to as the osculating elements at t_0. The velocity of the disturbed satellite at this time in its osculating ellipse is equal to its velocity in the actual orbit.

At a future time t_1, the elements will change due to the presence of R. For instance, a_0 will be changed to a_1, and the quantities $(a_1 - a_0), (e_1 - e_0)$, etc. are called perturbations in the elements. Corresponding to the perturbations in the elements are perturbations in the coordinates and velocity components. There are, however, at least two reasons for using orbital elements rather than coordinates to describe the motion of the satellite. First, the elements do not exhibit a variability of anomalistic motion that the coordinates do, therefore any variation can be attributed directly to the perturbing forces. Second, the elements give a clearer geometric picture than do the coordinates; hence the effect of the perturbation on the orbit can be seen immediately.

There are various kinds of disturbances that an orbit can experience, the severity of each is usually due to the altitude of the satellite. It is the purpose of

this chapter to give a qualitative description of the most important disturbances, and to relate the relative magnitudes of each.

B. THE EARTH'S GRAVITATIONAL FIELD

The two-body problem assumes that the earth is a sphere, while in reality the earth is flattened somewhat at the poles and bulges correspondingly at the equator. Such a shape is called an oblate spheroid. In the science of geodesy, it has been useful to define a reference ellipsoid as a mathematical surface which is an idealized approximation to the earth's actual surface. The study of satellite orbits has established a flattening of the terrestrial ellipsoid as 1/298.24, which corresponds to a difference between the equatorial and polar radii of 21.4 kilometers.

Another surface commonly used in geodesy is the geoid, which is the equipotential surface that coincides on the average with mean sea level in the oceans. The geoid is everywhere perpendicular to a plumb-line since gravity is always normal to its surface. Before the advent of artificial satellites, it was generally accepted that except for relatively small variations resulting from the presence of mountains or deep valleys, the geoid could be regarded as approximately an ellipsoid. Now, the surface of constant gravitation can be more accurately portrayed by representing the potential as a series of quantities known as spherical harmonics, each of which makes a contribution, positive, negative, or zero, to the total. The contribution of any harmonic is determined by a factor, usually represented by the symbol J and commonly referred to as the value of that harmonic. These J values for a planet's gravitational field can be determined from observations of a satellite orbit and they can be related to the shape of the geoid.

A large number of harmonics may be required to precisely represent a planet's gravitational field, but in practice the higher harmonics make such a small contribution

that they can be neglected, at least to a first approximation. The zonal harmonic J_0 expresses the overall size of the geoid, while J_1, the first degree harmonic determines the center point of the geoid in the north-south direction. The other harmonics represent deviations from the spherical shape as shown by Figure 4.1. It is seen that the contributions from the even harmonics are symmetrical about the equator, while the odd harmonics corresponds to contributions that are asymmetrical. The degree of the harmonic gives the number of undulations in the shape of the surface.

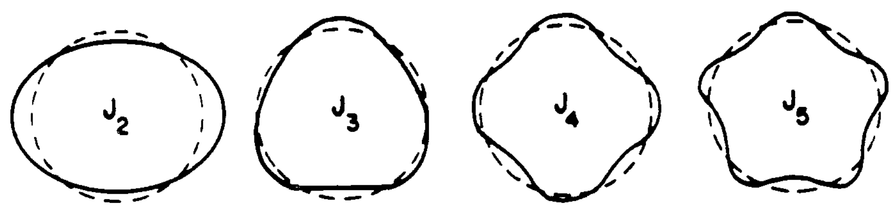

Figure 4.1: Qualitative Representation Of The Harmonics Of The Geoid

The results given thus far consider only the zonal harmonics, which are independent of longitude. The tesseral harmonics give east-west deviations from symmetry. Satellite observations of the tesseral harmonics have led to the conclusion that the equator of the earth's geoid is slightly elliptical, the difference between the longest and shortest axes being about 400 meters.

The following tables from Kozai [Ref. 29] gives representative values for the coefficients of the earth's zonal and tesseral harmonics.

TABLE 4.1: ZONAL HARMONICS

n	$J_n \times 10^6$			n	$J_n \times 10^6$		
2	1082.48	$\pm$	.04	6	.39	$\pm$	.09
3	−2.57	$\pm$	.01	7	−.47	$\pm$	.02
4	−1.84	$\pm$	.09	8	−.02	$\pm$	.07
5	−.06	$\pm$	.02	9	.11	$\pm$	.03

TABLE 4.2: TESSERAL HARMONICS

n	m	$J_n^m \times 10^6$			ϕ_{nm}		
2	2	2.32	$\pm$	.30	−37°.5	$\pm$	5°.6
3	1	3.95	$\pm$	.36	22°.0	$\pm$	11°.0
3	2	.41	$\pm$	.36	31°.0	$\pm$	14°.0
3	3	1.91	$\pm$	.29	51°.3	$\pm$	2°.9
4	1	2.64	$\pm$	.44	163°.5	$\pm$	6°.5
4	2	.17	$\pm$	.06	54°.0	$\pm$	11°.0
4	3	.05	$\pm$	.04	−13°.0	$\pm$	19°.0

C. ATMOSPHERIC DRAG

Atmospheric drag is the most complex and the most difficult of the artificial satellite perturbations. The complexity arises from the fact that an exact force law is not known and the atmosphere is variable. This variability results from the fact that the atmosphere is not spherically symmetric and that lunisolar tides, diurnal heating, strength of the solar wind, etc., all effect atmospheric density. In addition, the atmosphere is moving with the rotating earth. Therefore this perturbation is difficult to model analytically or numerically.

For most Earth satellites, atmospheric drag removes the satellite's energy, thereby causing it to drop in altitude and thus increase its speed. The increased speed and the lower altitude increases the drag force with the result that the satellite spirals into Earth.

Satellite drag is measured along the same direction as the velocity vector of the satellite. The formula used is

$$\mathbf{F}_D = \frac{-1}{2} C_D A \rho |\mathbf{v} - \mathbf{v_a}| (\mathbf{v_a} - \mathbf{v})$$

where $\mathbf{v}$ is the satellite's inertial velocity, $\mathbf{v_a}$ is the inertial velocity of the atmosphere, ρ is the atmospheric density, A is the effective cross-sectional area of the satellite, and C_D is the drag coefficient, $C_D \approx 2$.

D. SOLAR RADIATION PRESSURE

In contrast to drag which is a tangential force, solar radiation pressure is radial. By solar radiation on a satellite is meant the net acceleration resulting from the interaction (i.e., absorption, reflection, or diffusion) of the sunlight with the surface of the satellite. In general, the illumination of a low altitude satellite orbit will be non-uniform due to the Earth's shadow, albedo, and atmospheric absorption. The magnitude of the solar radiation pressure is given by

$$\frac{A P}{4 \pi m c D^2}$$

where A is the effective cross-sectional area of the satellite, P is the total radiated solar power, m is the satellite's mass, c is the speed of light, and D is the satellite-Sun distance.

It can be seen that the perturbing effect of solar radiation particularly effects satellites with a large area to mass ratio. Such was the case of the satellite Echo I, which when inflated was a sphere with a total exposed area of about 31,000 square feet and weighed only 135 pounds. Its initial orbit was approximately circular with an altitude of 1000 miles, but the pressure of solar radiation brought down the perigee to 600 miles at times.

A. OTHER PERTURBING FORCES

Gravitational perturbations due to other celestial bodies (the Moon, the Sun, and the planets) are caused by the differences between the force on the Earth and that on the satellite or the "tidal" force. Perturbing forces from the Sun and Moon become significant as the altitude of the satellite increases. Planetary perturbations are very small, with Venus and Jupiter providing the largest contributions.

Since classical mechanics is used in this analysis, relativistic corrections may be regarded as a small perturbation to the motion. Other minor perturbing forces include atmospheric lift and electromagnetic forces.

The following table from Milani [Ref 30] gives the magnitudes of various perturbations on three different satellites:

TABLE 4.3: ACCELERATIONS ON SPACECRAFT AT VARIOUS ALTITUDES (cm/sec^2)

Cause	Geosynchronous satellite a = 42,160 KM A/M = .1 cm^2/g	LAGEOS 12,270 .007	SEASAT 7,100 .2
Earth's monopole	2.2×10^{1}	2.8×10^{2}	7.9×10^{2}
Earth's oblateness	7.4×10^{-4}	1.0×10^{-1}	9.3×10^{-1}
Higher harmonics	4.5×10^{-10}	8.8×10^{-6}	7.0×10^{-4}
Moon	7.3×10^{-4}	2.1×10^{-4}	1.3×10^{-4}
Sun	3.3×10^{-4}	9.6×10^{-5}	5.6×10^{-5}
Venus	4.3×10^{-8}	1.3×10^{-8}	7.3×10^{-9}
General Relativity	2.3×10^{-9}	9.3×10^{-8}	4.9×10^{-7}
Air Drag	0 (?)	3.0×10^{-10}	2.0×10^{-5}
Solar Radiation	4.6×10^{-6}	3.2×10^{-7}	9.2×10^{-6}
Earth's albedo	4.2×10^{-8}	3.4×10^{-8}	3.0×10^{-6}

V. DEVELOPMENT OF THE EQUATIONS OF MOTION

A. PRELIMINARIES

1. Overview

In this chapter, the groundwork will be laid for solving the equations of motion for a satellite about an oblate planet. To begin, a discussion of the assumptions made in the analysis will be given. Second, the special coordinate system, which was introduced by King-Hele [Ref. 26] and refined by Brenner and Latta [Ref. 27] will be developed, followed by a derivation of the relationships among the astronomical angles of the coordinate system and the angles of a spherical coordinate system. Next, an expansion for the potential of a planet modeled as an oblate spheroid will be derived. Finally, the equations will be transformed so that they can be solved by an ordinary perturbation method.

2. Assumptions and Limitations

The basis for the assumptions made in this analysis is the requirement that a solution to the equations of motion for a satellite orbiting a planet be accurate to within a relative error $\leq 10^{-6}$. Therefore, all perturbative forces that are of magnitude 10^{-6} or smaller compared to 1 may be neglected. This requirement then allows one to model the earth as an oblate spheroid with an axially symmetric gravitational potential. This is to say that all zonal harmonics except the second in the expanded potential may be neglected. In addition, all coefficients of the tesseral harmonics are small enough to neglect.

The assumption of an axially symmetric potential is a reasonable assumption if the earth is used as an example of an oblate planet. As was suggested in Chapter IV, if one normalizes the expanded geopotential of the earth such that the contribution from the central force ($1/r^2$) is 1, then the contribution from the zonal harmonic for the oblateness contribution is $\approx 10^{-3}$, and all other harmonics (zonal and tesseral) are $\leq 10^{-6}$.

The gravitational effects of the Sun and Moon are also neglected. This assumption will remain valid out to a distance of at least 6,000 kilometers above the Earth, where the oblateness perturbation is still about 1000 times larger than the attraction of the Sun. However, at altitudes near geosynchronous (35,800 km) the perturbative forces of the Sun and Moon are nearly equal to that of oblateness, and therefore would have to be considered.

The most important limitation to the analysis is the neglect of atmospheric drag. For high altitude satellites with small eccentricity, the neglect of air drag is unimportant; however, for very low altitude satellites the effect of air drag would begin to dominate the oblateness force after a number of revolutions. Also, high altitude satellites with highly eccentric orbits would be greatly affected by drag as they pass through perigee. For these particular cases, the desired accuracy for 1000 revolutions would not be achievable; however, in many cases the solution could be accurate for a few orbits. As was discussed in Chapter IV, air drag remains the most difficult perturbation to model. Since an accurate geopotential model allows for a better determination of fluctuations in drag of the atmosphere through which the satellite is moving, the analysis conducted here is valuable even when air drag becomes the dominating factor.

Also neglected in this analysis, are the remaining perturbative effects mentioned in Chapter IV. Solar radiation pressure, the Earth's albedo radiation

pressure, and the general relativistic correction to Newton's law of gravity are all neglected since their relative contributions are small.

The final limitation to the analysis is the assumption that the differential equations which describe the motion of the satellite have a solution asymptotic to the form specified in the analysis. As long as the true solution does not depart from the form specified, the analysis is valid.

3. Order of Magnitudes

Throughout this analysis, reference will be made to the relative magnitudes of the perturbing forces; hence it is important to establish a convention for orders of magnitudes. The approach established by King-Hele [Ref 26] will be followed here with the exception that a small eccentricity is not assumed. Here, $J \approx J_2$, the coefficient of the second zonal harmonic in the potential is used as the basic small unit. Mathematically,

$$f(J, \theta) = O(J^m \theta^n)$$

means that there exists for all

$$0 \leq J \leq 10^{-3} \text{ and } 0 \leq \theta \leq (2\pi)10^3$$

a K independent of J and θ such that

$$|f| < K J^m \theta^n$$

The following terms are defined:

Zero order	$\equiv$	$O(1)$	$\lesssim$	1.0
First order	$\equiv$	$O(J), O(J^2\theta), O(J^3\theta^2), \ldots$	$\lesssim$	10^{-3}
Second order	$\equiv$	$O(J^2), O(J^3\theta), O(J^4\theta^2), \ldots$	$\lesssim$	10^{-6}

B. THE COORDINATE SYSTEM

Figure 5.1 is an inertial reference system of spherical coordinates with the prime direction pointing toward the vernal equinox at epoch 1900.0. The equatorial plane, latitude (δ), and longitude (ϕ), as shown in the figure have their usual meanings.

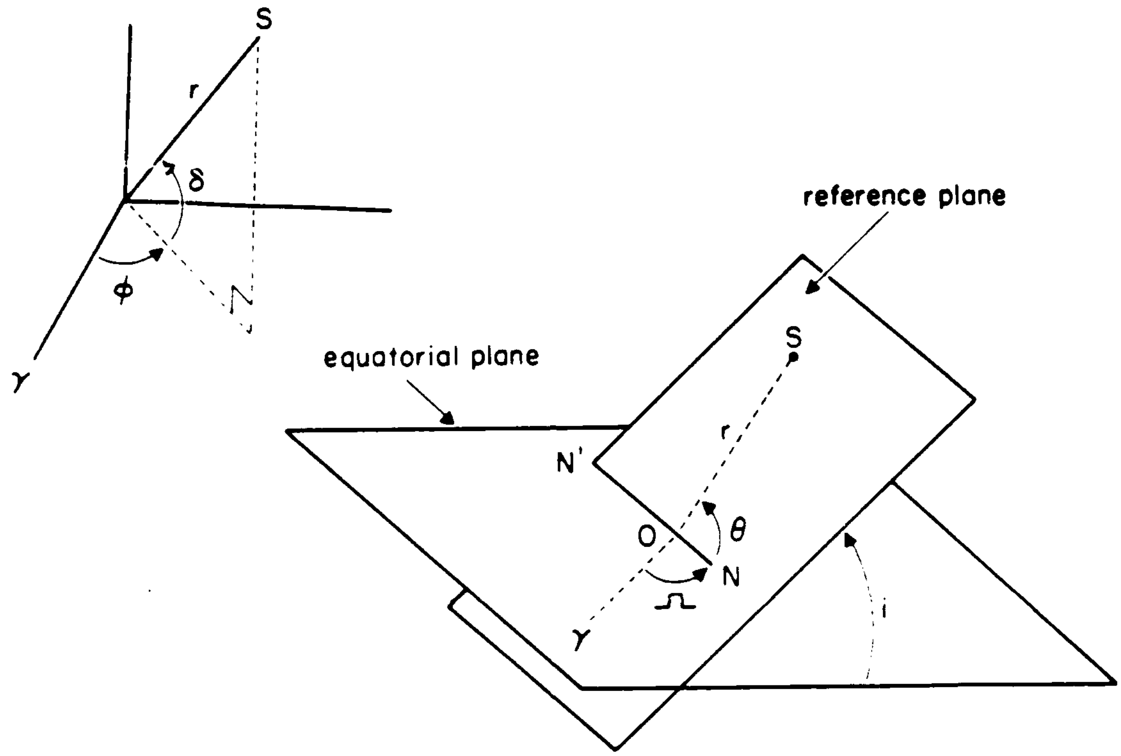

Figure 5.1: The Reference System

As was demonstrated in Chapter III, the path of a satellite of a strictly spherical planet lies entirely in a fixed plane, and the motion of the satellite is described by the solution to the two-body problem. With the angular momentum vector fixed in space, the intersection of the orbit plane and the equatorial plane describe the fixed angle Ω measured from the prime direction to the intersection (or node). The longitude of the ascending node (Ω), the inclination (i), and the argument of perigee (ω), fix the orbit plane in the coordinate system.

The effect of the oblateness perturbation is, in general, to move the satellite out of the original two-body orbital plane. One specific effect is to rotate the orbit plane about the planet in the opposite direction to the satellite's motion. The physics of this motion is easily demonstrated in Figure 5.2. Oblateness is represented by an additional mass about the equator of the planet. The radius vector $\mathbf{r}$ and the satellite S lie in a plane which is named here the "reference plane". If there were no equatorial bulge, the direction of the gravitational force would coincide with the radius vector and the angular momentum vector would remain in a constant direction normal to the plane. The equatorial bulge; however, adds a component of force that does not lie along $\mathbf{r}$. This additional force adds a torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. The direction of $\boldsymbol{\tau}$ is into the paper at S for a prograde orbit. Therefore as the satellite orbits, the angular momentum vector rotates about the Z axis. The reference plane also rotates, and the rotation is measured by the change in Ω.

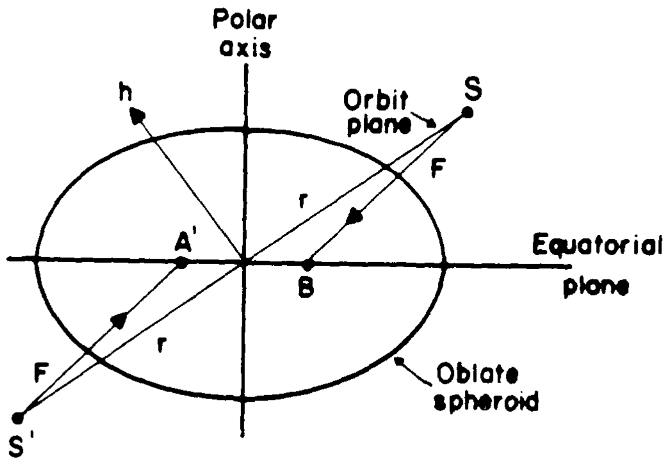

Figure 5.2: Rotation of the Reference Plane

The line NON' in Figure 5.1 is referred to as the "line of nodes", and the ascending node describes the point where the satellite enters the northern hemi-

sphere. For the two-body problem with a non-rotating plane, the line of nodes is a fixed reference line. Now, for the rotating plane, when the satellite is actually at a node, the line of nodes passes through the satellite, but at other times the definition "line of nodes" is an arbitrary one, and the angle Ω is simply given as a function of time.

Another motion of the reference plane that is a result of the oblateness perturbation is an in-plane rotation called precession. As was demonstrated in Chapter III, a particle executing bounded, non-circular motion in a central force field will always have a radial distance from the force center that is bounded by $r_{max} \leq r \leq r_{min}$. That is, r is bounded by the apsidal distance. Such an orbit is called a closed orbit and it is characterized by the fact that all apsidal angles are equal. For instance, the apsidal angle is π for elliptic motion. But if the radial dependence deviates slightly from $1/r^2$, then the apsides will precess or rotate slowly in the plane of motion (Figure 5.3). This motion is analogous to the slow rotation of elliptic motion of a two dimensional harmonic oscillator whose natural frequencies for each dimension are almost equal. The rotation of the line of apsides is also known as the precession of perigee since the value for θ for which r is a minimum varies in time as the apsides rotate.

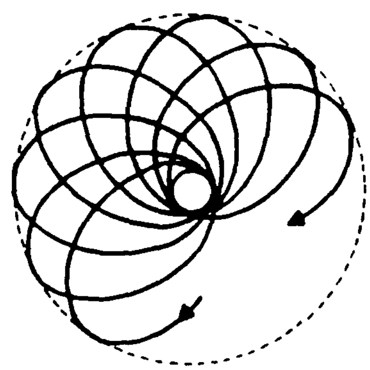

Figure 5.3: Rotation of the Line of Apsides

For a satellite orbiting an oblate planet the rotation of the line of apsides is opposite the direction of satellite motion for inclinations less than $\sin^{-1}\sqrt{4/5}$ and in the same direction as the satellite for inclinations greater than $\sin^{-1}\sqrt{4/5}$. If the inclination is exactly $\sin^{-1}\sqrt{4/5}$ then there is no rotation of the line of apsides, and perigee remains constant. This inclination is known in Astrodynamics as the "critical inclination". A detailed discussion of the critical inclination is contained in Appendix C.

A final motion of the reference plane caused by the oblateness perturbation is the periodic variation of the inclination about the initial inclination i_0.

C. ANGLE RELATIONS

As a preliminary to deriving the equations of motion, it is necessary that the relationships among the spherical coordinates and the angles describing the reference plane be established.

Reference is made to Figure 5.4, where $\mathbf{i}, \mathbf{i}', \mathbf{j}$, and $\mathbf{j}'$ define the equatorial plane. Let $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ denote three unit vectors, where $\mathbf{b}$ and $\mathbf{c}$ are in the reference plane. $\mathbf{c}$ points to the initial point $\theta = \pi/2$ in the reference plane where θ is measured from the line of nodes. $\mathbf{b}$ points to the position of the satellite, S. $\mathbf{a}$ is perpendicular to the reference plane and therefore points in the same direction as the angular momentum vector $\mathbf{h}$. $\mathbf{a}$ and $\mathbf{c}$ are both perpendicular to the line of nodes. Then

$$\mathbf{b} = \cos\phi\cos\delta\,\mathbf{i} + \sin\phi\cos\delta\,\mathbf{j} + \sin\delta\,\mathbf{k}.$$

Now measure $\mathbf{b}$ from the line of nodes.

$$\mathbf{b} = \cos(\phi - \Omega)\cos\delta\,\mathbf{i}' + \sin(\phi - \Omega)\cos\delta\,\mathbf{j}' + \sin\delta\,\mathbf{k}.$$

a and c are therefore

$$\mathbf{a} = -\sin i\mathbf{j}' + \cos i\mathbf{k}$$
$$\mathbf{c} = \cos i\,\mathbf{j}' + \sin i\,\mathbf{k}.$$

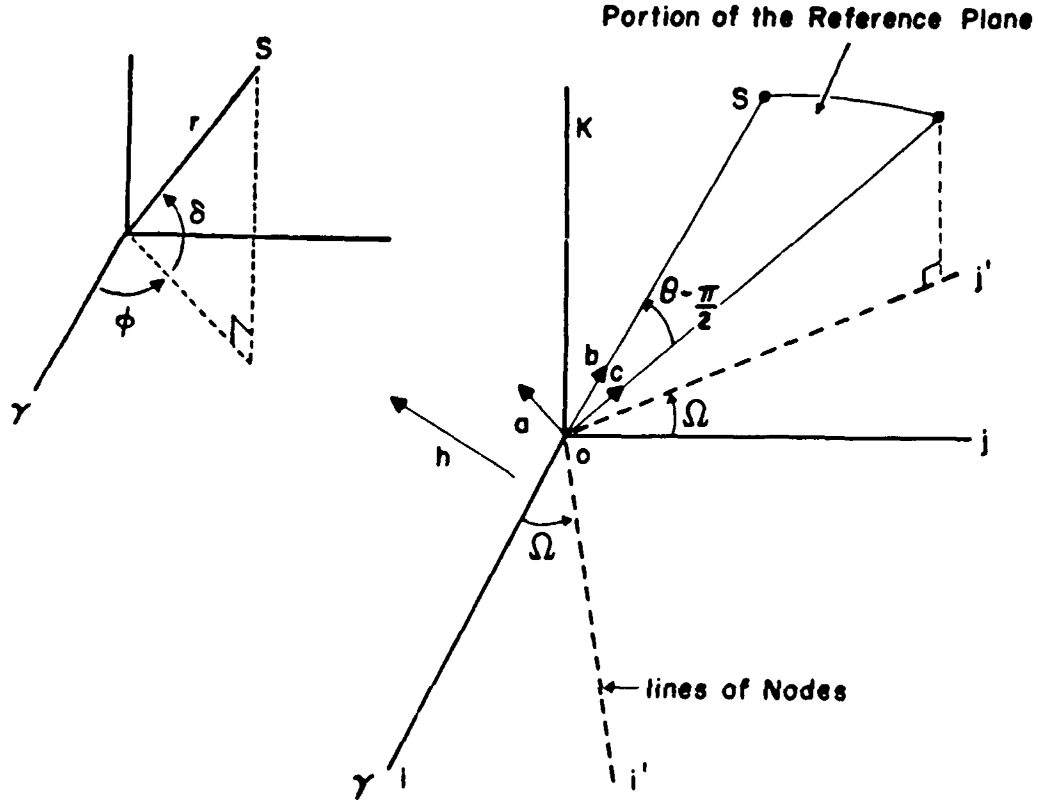

Figure 5.4: Angle Relations

Since a and b are perpendicular, $\mathbf{a} \cdot \mathbf{b} = 0$, therefore

$$\tan\delta = \tan i \sin(\phi - \Omega). \quad (5.1)$$

Continuing

$$\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}|\ |\mathbf{c}|\ \cos(\theta - \pi/2)$$

results in

$$\sin(\phi - \Omega)\cos\delta\ \cos i + \sin\delta\ \sin i = \sin\theta.$$

And using the relation from equation (5.1) results in

$$\sin\delta = \sin\theta \sin i. \tag{5.2}$$

Finally, from $\mathbf{c} \times \mathbf{b}$, the following relationship is obtained

$$\cos\delta = \cos\theta \sec(\theta - \Omega). \tag{5.3}$$

(5.1), (5.2), and (5.3) are the required angle relations.

D. THE POTENTIAL

In Chapter III, a simplified gravitational potential GM/r was used with the assumption that the attracting body was spherically symmetric. This simplified potential caused the satellite to move in conic orbits. As has been stated, the planets are not spherically symmetric but are bulged at the equator, flattened at the poles, and are generally asymmetric. An expanded expression for the gravitational potential will be developed in this section. The final expression for the potential is subject to the assumption made in Section B of this chapter that the planet about which the satellite revolves is approximately an oblate spheroid.

Now, regardless of the nature of an attracting body, the potential V must satisfy one of the following differential equations. For regions within the attracting matter

$$\nabla^2 V = 4\pi\rho G \text{ where } \rho \text{ is the density} \tag{5.4}$$

which is Poisson's equation.

For regions outside attracting matter

$$\nabla^2 V = 0 \tag{5.5}$$

which is Laplace's equation.

This discussion will be restricted to Laplace's equation which, if V is written as a function of the spherical coordinates (r, θ, ϕ), may be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\cos\delta}\frac{\partial}{\partial\delta}\left(\cos\delta\frac{\partial V}{\partial\delta}\right) + \frac{1}{r^2\cos^2\delta}\frac{\partial^2 V}{\partial\phi^2} = 0. \tag{5.6}$$

Any conservative force field $\mathbf{F}$ can be written as the gradient of a potential. Equations (5.5) and (5.6) state that div (grad V) is zero. The assumption that $\mathbf{F}$ is rotationally symmetric means that $\partial V/\partial\phi$ is zero. Therefore a solution of (5.6) is sought that is a product of a function of r alone and a function of δ alone:

$$V(r,\delta) = f_1(r)\cdot f_2(\delta).$$

By multiplying (5.6) by r^2, dividing by $f_1(r)f_2(\delta)$, and rearranging terms, the following result is obtained.

$$\frac{1}{f_1}\frac{d}{dr}\left(r^2\frac{df_1}{dr}\right) = -\frac{1}{f_2\cos\delta}\frac{\partial}{\partial\delta}\left(\cos\delta\frac{df_2}{d\delta}\right).$$

Since the left side is a function of r alone and the right side is a function of δ alone, both members are equal to a constant, say $n(n+1)$. f_1 must satisfy

$$\frac{d}{dr}\left(r^2\frac{df_1}{dr}\right) = n(n+1)f_1$$

which has the solution

$$fr_1 = Ar^{-n-1} + Br^n.$$

It is desired that this function be zero at $r = \infty$, and that it be analytic and single-valued there. Therefore n is chosen to be an integer greater than zero, and B most equal zero. Therefore

$$f_1 = Ar^{-n-1}.$$

The equation for f_2 is

$$\frac{d}{d\delta}\left(\cos\delta\frac{df_2}{d\delta}\right) + n(n+1)\cos\delta f_2 = 0$$

which may be rewritten as

$$\frac{d}{d\sin\delta}\left[(1-\sin^2\delta)\frac{df_2}{d\sin\delta}\right]+n(n+1)f_2=0. \tag{5.7}$$

This is known as Legendre's equation. Using the method of Frobenius, a series solution to Legrendre's equation is found to be

$$f_2 = CP_n(\sin\delta).$$

Therefore the solution to (5.6) may finally be written as

$$V(r,\delta) = Ar^{-n-1}P_n(\sin\delta)$$

where C is chosen as 1.

There are many ways to consider the polynomials P_n. Rodrique's formula for $P_n(\sin\delta)$ is

$$P_n(\sin\delta) = \frac{1}{2^n n!}\frac{d^n}{d(\sin\delta)^n}(\sin^2\delta - 1)^n.$$

so that

$$P_0 = 1, \quad P_1 = \sin\delta, \quad P_2 = \frac{1}{2}(3\sin^2\delta - 1), \quad \text{etc.}$$

The general solution of (5.6) is then written as

$$V(r,\delta) = \sum_{n=0}^{\infty} A_n r^{-n-1} P_n(\sin\delta). \tag{5.8}$$

The above mathematical derivation of an expression for the potential may be enhanced by a more physical formulation. Referring to Figure 5.5, let a unit mass m be placed at a point P which is a distance r from the center of mass of a bounded distribution of total mass M. Let dm be an element of the mass at a distance ℓ from O. Then the potential at P due to dm is

$$dV = \frac{-K^2 dm}{(\ell^2 + r^2 - 2r\ell\cos\theta)^{1/2}}$$

and the total potential is

$$V = -K^2 \int_M \frac{dm}{(\ell^2 + r^2 - 2r\ell\cos\theta)^{1/2}}. \tag{5.9}$$

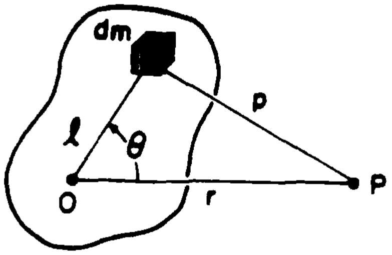

Figure 5.5: Potential at an Exterior Point Due to an Irregular Mass

The denominator of the right side of (5.9) may be expanded such that

$$(\ell^2 + r^2 - 2r\ell\cos\theta)^{-1/2} = \frac{1}{r}\sum_{n=1}^{\infty}\left(\frac{\ell}{r}\right)^n P_n(\sin\delta)$$

where $\delta = \pi/2 - \theta$.

Therefore the following may be written

$$V = \frac{-K^2}{r}\int_M dm - \frac{K^2}{r^2}\int_M \ell\sin\delta\, dm + \frac{K^2}{2r^3}\int_M \ell^2 dm$$
$$-\frac{3}{2}\frac{K^2}{r^3}\int_M \ell^2\sin^2\delta\, dm + \ldots$$

A description of the physical significance of each of the above integrals can be made. The first integral is the total mass, the second is the first moment about an axis through O perpendicular to OP and is zero when the origin is chosen as the center of mass. The third integral is the moment of inertia about the origin, and the last integral may be written as

$$\int_M (\ell^2 - \ell^2\cos^2\delta)dm = I_0 - I$$

where I_0 is the moment of inertia about the origin and I is the moment of inertia about the line OP.

The potential is then

$$V = -\frac{K^2 M}{r} - \frac{K^2}{2r^3}(2I_0 - 3I).$$

Now the first term is the potential due to a homogeneous solid sphere. The second term arises from the departure of the mass M from spherical shape.

With the physical description established, (5.8) may be written as

$$V = -\frac{GM}{r}\left[1 + J_2\left(\frac{R}{r}\right)^2\left(\frac{1}{2} - \frac{3}{2}\sin^2\delta\right) + \ldots\right] \tag{5.10}$$

Note that only P_0 and P_2 remain.

An advantage of (5.10) is that it can immediately be written down once axial symmetry has been assumed, and any suitable experiment (such as an orbit of a satellite about the Earth) can be used to determine J_2 (or J_3, J_4, etc.; for general potential).

E. THE EQUATIONS OF MOTION

Referring again to Figure 5.1, the components of the velocity of a satellite in the coordinate system are:

$$v = v_r + v_\phi + v_\delta$$

or

$$v = \frac{dr}{dt} + r\frac{d\phi}{dt}\cos\delta + r\frac{d\delta}{dt}.$$

An expression for the kinetic energy (T) is

$$2T = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\delta}{dt}\right)^2 + r^2\cos^2\delta\left(\frac{d\phi}{dt}\right)^2. \tag{5.11}$$

The equations of motion may be written using Lagrange's equation

$$\frac{d}{dt}\frac{\partial(T-V)}{\partial \dot{q}} - \frac{\partial(T-V)}{\partial q} = 0 \qquad q \equiv r, \delta, \text{ or } \phi \tag{5.12}$$

where $\dot{q} = \frac{dq}{dt}$, T is the kinetic energy and V is the potential energy.

Applying (5.12) to the expression for T (5.11) and V (5.10) results in

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\delta}{dt}\right)^2 - r\cos^2\delta\left(\frac{d\phi}{dt}\right)^2 = -\frac{\partial V}{\partial r} \tag{5.13}$$

$$\frac{d}{dt}\left(r^2\frac{d\delta}{dt}\right) + r^2\sin\delta\cos\delta\left(\frac{d\phi}{dt}\right)^2 = -\frac{\partial V}{\partial \delta} \tag{5.14}$$

$$\frac{d}{dt}\left(r^2\cos^2\delta\frac{d\phi}{dt}\right) = 0 \tag{5.15}$$

The last of these equations gives an integral which can be used to eliminate time from the system, i.e.,

$$r^2\cos^2\delta\frac{d\phi}{dt} = \bar{h}\cos i_0.$$

Let $r = \bar{p}/u$ then

$$\frac{d}{dt} = \frac{\bar{h}\cos i_0 u^2 sec^2\delta}{\bar{p}^2}\frac{d}{d\phi}. \tag{5.16}$$

Starting with equation (5.14) and applying (5.16) to the first term results in

$$\frac{d}{dt}\left(r^2\frac{d\delta}{dt}\right) = \frac{d}{dt}\left(\frac{r^2\bar{h}\cos^2 i_0 sec^2\delta}{\bar{p}^2}\right)\frac{d\delta}{d\phi},$$

and continuing to apply (5.16) so that (5.14) finally becomes

$$\frac{d^2\tan\delta}{d\phi^2} + \tan\delta = -\frac{\partial V}{\partial\delta}\frac{r^2\cos^2\delta}{\bar{h}^2\cos^2 i_0}$$

or

$$\frac{d^2\tan\delta}{d\phi^2} + \tan\delta = \frac{\sin\delta\cos^3\delta}{\cos^2 i_0}(2Ju) \tag{5.17}$$

where

$$J = \frac{3}{2}\left(J_2 \frac{R^2}{\bar{p}^2}\right)$$

$$\bar{p} = \bar{h}^2/GM.$$

It is desired to get the left side of (5.17) in terms of the independent variable θ. To begin, the right side of the angle relationship (5.1),

$$\tan\delta = \tan i \sin(\phi - \Omega),$$

is substituted into (5.17).

Then using (5.3)

$$\cos\delta = \cos\theta \sec(\phi - \Omega),$$

(5.17) becomes

$$\tan\delta\left[\left(1 - \frac{\Omega'}{\phi'}\right)^2 - 1\right] - \frac{2\sec^2 i\cos\theta}{\cos\delta}\left(1 - \frac{\Omega'}{\phi'}\right)\frac{i'}{\phi'} - \frac{\tan i\cos\theta}{\cos\delta}\left(\frac{\Omega'\phi'' - \Omega''}{\phi'^2}\right)$$

$$-2\sec^2 i\tan\delta\left(\frac{i'}{\phi'}\right)^2 - \frac{\tan\delta}{\sin i\cos i}\left(\frac{i'' - i'\phi''}{\phi'^2}\right)$$

$$= \frac{\sin\delta\cos^3\delta}{\cos^2 i_0}(2J\,u), \tag{5.18}$$

where primes denote differentiation with respect to θ.

Later, ϕ will be eliminated from (5.18), but first the remaining equation of motion (5.13) will be rewritten in a form like (5.18).

To rewrite equation (5.13) in a useful form, the independent variable is again changed by use of (5.16). Multiplying the expression by $(\phi')^2$ and noting that

$$(\tan^2\delta)' = 2\tan\delta\sec^2\delta\frac{d\delta}{d\theta}$$

results in:

$$u'' + u\left[\phi'^2\cos^2\delta + \frac{(\tan\delta)'^2}{(\sec^4\delta)}\right] + \frac{u'(\tan^2\delta)'}{\sec^2\delta} - \frac{u'\phi''}{\phi'}$$

$$= \frac{\phi'\cos^4\delta}{\bar{h}^2u^2}\frac{\partial V}{\partial r}. \tag{5.19}$$

Evaluating $\frac{\partial V}{\partial r}$ and multiplying (5.19) by $\frac{\cos^2 i}{\cos^4\delta}\left(\frac{1}{\phi'}\right)^2$ results in:

$$\begin{aligned} u'' + u &= \frac{\cos^2 i}{\cos^2 i_0}[1 + J\,u^2(1 - 3\sin^2\delta)] + u''\left[1 - \frac{1}{\left(1 + \Omega'\frac{\cos^2\delta}{\cos i} - i'\cos\theta\sin\theta\tan i\right)^2}\right] \\ &- \frac{u'\cos^2 i\sec^4\delta}{\phi'^2}\left[2\sin\delta\cos\delta(\tan\delta)' - \frac{\phi''}{\phi'}\right] \\ &- u\left[\cos^2 i\sec^2\delta + \frac{(\tan\delta)'^2\cos^2 i}{\phi'^2} - 1\right]. \end{aligned} \tag{5.20}$$

Equations (5.18) and (5.20) are the equations of motion. Before proceeding to solve these equations, it is necessary to remove ϕ. Combining equations (5.1) - (5.3) results in the following expression:

$$\tan(\phi - \Omega) = \cos i\tan\theta.$$

Differentiating this expression with respect to θ results in

$$\phi' = \frac{\cos i}{\cos^2\delta} + \Omega' - \frac{i'\sin\theta\cos\theta\sin i}{\cos^2\delta} \tag{5.21}$$

or

$$\frac{1}{\phi'} = \frac{\cos^2\delta}{\cos i} + \frac{1}{1 + \Omega'\frac{\cos^2\delta}{\cos i} - i'\sin\theta\cos\theta\tan i} \tag{5.22}$$

From (5.2), $\sin^2\delta = (\sin\theta\sin i)^2$ or

$$\cos^2\delta = 1 - (\sin\theta\sin i)^2.$$

Substitution of this expression plus (5.21) and (5.22) in (5.18) and (5.20) results in completely general equations of motion in terms of the dependent variables: i, Ω, y; and the independent variable θ.

$$
\begin{aligned}
& \frac{6 \sin i \left(\frac{di}{d\theta}\right)^2 \sin^3 \theta}{\cos i} - \frac{3 \sin i \frac{di}{d\theta} \frac{d^2 i}{d\theta^2} \cos \theta \sin^2 \theta}{\cos i} \\
+ & \frac{3 \sin^2 i \frac{di}{d\theta} \frac{d^2\Omega}{d\theta^2} \cos^2 \theta \sin \theta}{\cos i} + \frac{6 \sin^3 i \left(\frac{d\Omega}{d\theta}\right)^2 \cos^2 \theta \sin \theta}{\cos i} \\
+ & \frac{3 \sin^2 i \frac{d^2 i}{d\theta^2} \frac{d\Omega}{d\theta} \cos^2 \theta \sin \theta}{\cos i} + \frac{2 \sin^2 i \frac{di}{d\theta} \frac{d^2\Omega}{d\theta^2} \sin \theta}{\cos i} \\
+ & 3 \cos i \frac{di}{d\theta} \frac{d^2\Omega}{d\theta^2} \sin \theta - \frac{2 \frac{di}{d\theta} \frac{d^2\Omega}{d\theta^2} \sin \theta}{\cos i} \\
+ & \frac{3 \sin^3 i \left(\frac{d\Omega}{d\theta}\right)^2 \sin \theta}{\cos i} + 6 \cos i \sin i \left(\frac{d\Omega}{d\theta}\right)^2 \sin \theta \\
- & \frac{3 \sin i \left(\frac{d\Omega}{d\theta}\right)^2 \sin \theta}{\cos i} + \frac{\sin^2 i \frac{d^2 i}{d\theta^2} \frac{d\Omega}{d\theta} \sin \theta}{\cos i} \\
+ & 3 \cos i \frac{d^2 i}{d\theta^2} \frac{d\Omega}{d\theta} \sin \theta - \frac{\frac{d^2 i}{d\theta^2} \frac{d\Omega}{d\theta} \sin \theta}{\cos i} - 2 \sin i \frac{d\Omega}{d\theta} \sin \theta \\
- & \frac{d^2 i}{d\theta^2} \sin \theta - \frac{6 \sin i \frac{di}{d\theta}^2 \sin \theta}{\cos i} - \frac{3 \sin^3 i \frac{d\Omega}{d\theta} \frac{d^2\Omega}{d\theta^2} \cos^3 \theta}{\cos i} \\
+ & \frac{12 \sin^2 i \frac{di}{d\theta} \frac{d\Omega}{d\theta} \cos^2 \theta}{\cos i} - 3 \cos i \sin i \frac{d\Omega}{d\theta} \frac{d^2\Omega}{d\theta^2} \cos \theta \\
+ & \sin i \frac{d^2\Omega}{d\theta^2} \cos \theta + \frac{2 \sin^2 i \frac{di}{d\theta} \frac{d\Omega}{d\theta} \cos \theta}{\cos i} \\
+ & 12 \cos i \frac{di}{d\theta} \frac{d\Omega}{d\theta} \cos \theta - \frac{8 \frac{di}{d\theta} \frac{d\Omega}{d\theta} \cos \theta}{\cos i} - 2 \frac{di}{d\theta} \cos \theta \\
= & \frac{2 \cos^3 i \sin i \, J \, u \sin \theta}{\cos^2 i_0^1}
\end{aligned}
\tag{5.23}
$$

and equation (5.20) is:

$$\frac{d^2u}{d\theta^2} + u = -\frac{3\sin^2 i\left(\frac{di}{d\theta}\right)^2 \frac{d^2u}{d\theta^2}\sin^2(2\theta)}{4\cos^2 i}$$

$$- \frac{3\sin^3 i\frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{d^2u}{d\theta^2}\sin^2\theta\sin(2\theta)}{\cos^2 i}$$

$$- \frac{\sin^3 i\frac{di}{d\theta}\frac{d^2\Omega}{d\theta^2}\frac{du}{d\theta}\sin^2\theta\sin(2\theta)}{2\cos^2 i}$$

$$- \frac{2\sin^3 i\frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\cos\theta\sin\theta\sin(2\theta)}{\cos^2 i}$$

$$- \frac{\sin^2 i\left(\frac{di}{d\theta}\right)^2 u\cos\theta\sin\theta\sin(2x)}{\cos^2 i} + \frac{\sin^2 i\left(\frac{di}{d\theta}\right)^2 \frac{du}{d\theta}\cos^2\theta\sin(2x)}{2\cos^2 i}$$

$$+ \frac{\sin^3 i\frac{di}{d\theta}\frac{d\Omega}{d\theta}u\cos^2\theta\sin(2\theta)}{\cos^2 i} + \frac{3\sin i\frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{d^2u}{d\theta^2}\sin(2\theta)}{\cos^2 i}$$

$$- \frac{\sin i\frac{di}{d\theta}\frac{d^2u}{d\theta^2}\sin(2\theta)}{\cos i} + \frac{\sin i\frac{di}{d\theta}\frac{d^2\Omega}{d\theta^2}\frac{du}{d\theta}\sin(2\theta)}{2\cos^2 i}$$

$$- \frac{3\sin^4 i\left(\frac{d\Omega}{d\theta}\right)^2\frac{d^2u}{d\theta^2}\sin^4\theta}{\cos^2 i} - \frac{3\sin^4 i\frac{d\Omega}{d\theta}\frac{d^2\Omega}{d\theta^2}\frac{du}{d\theta}\sin^4\theta}{\cos^2 i}$$

$$- \frac{2\sin^3\theta\frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\sin^4\theta}{\cos^2 i} + \frac{\sin^2 i\frac{di}{d\theta}\frac{d^2i}{d\theta^2}\frac{du}{d\theta}\sin^4\theta}{\cos^2 i}$$

$$- \frac{2\sin^3 i\frac{di}{d\theta}\frac{d^2\Omega}{d\theta^2}\frac{du}{d\theta}\cos\theta\sin^3\theta}{\cos^2 i}$$

$$- \frac{6\sin^4 i\left(\frac{d\Omega}{d\theta}\right)^2\frac{du}{d\theta}\cos\theta\sin^3\theta}{\cos^2 i}$$

$$- \frac{2\sin^3 i\frac{d^2i}{d\theta^2}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\cos\theta\sin^3\theta}{\cos^2 i}$$

$$- \frac{2\sin^2 i\left(\frac{di}{d\theta}\right)^2\frac{du}{d\theta}\cos\theta\sin^3\theta}{\cos^2 i}$$

$$- \frac{9\sin^3 i\frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\cos^2\theta\sin^2\theta}{\cos^2 i}$$

$$- \frac{2\sin^2 i\frac{di}{d\theta}\frac{d^2i}{d\theta^2}\frac{du}{d\theta}\cos^2\theta\sin^2\theta}{\cos^2 i} - \frac{\sin^2 i\frac{di}{d\theta}^2 u\cos^2\theta\sin^2\theta}{\cos^2 i}$$

$$+ \frac{6\sin^2 i\left(\frac{d\Omega}{d\theta}\right)^2\frac{d^2u}{d\theta^2}\sin^2\theta}{\cos^2 i} - \frac{2\sin^2 i\frac{d\Omega}{d\theta}\frac{d^2u}{d\theta^2}\sin^2\theta}{\cos i}$$

$$+ \frac{6\sin^2 i\frac{d\Omega}{d\theta}\frac{d^2\Omega}{d\theta^2}\frac{du}{d\theta}\sin^2\theta}{\cos^2 i} - \frac{\sin^2 i\frac{d^2\Omega}{d\theta^2}\frac{du}{d\theta}\sin^2\theta}{\cos i}$$

$$+ \frac{2\sin^3 i\frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\sin^2\theta}{\cos^2 i} + \frac{2\sin i\frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\sin^2\theta}{\cos^2 i}$$

$$- \sin i\frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\sin^2\theta - \frac{\frac{di}{d\theta}\frac{d^2i}{d\theta^2}\frac{du}{d\theta}\sin^2\theta}{\cos i}$$

$$+ \frac{di}{d\theta}\frac{d^2i}{d\theta^2}\frac{du}{d\theta}\sin^2\theta - \frac{\sin\frac{di}{d\theta}\frac{du}{d\theta}\sin^2\theta}{\cos i}$$

$$- \frac{3\cos^2 i\sin^2 i\,J\,u^2\sin^2\theta}{\cos^2 i_0} - \frac{di^2}{d\theta}u\sin^2\theta$$

$$+ \frac{\sin^3 i\frac{d^2i}{d\theta^2}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\cos^3\theta\sin\theta}{\cos^2 i}$$

$$- \frac{8\sin^2 i\left(\frac{di}{d\theta}\right)^2\frac{du}{d\theta}\cos^3\theta\sin\theta}{\cos^2 i} - \frac{\left(\frac{di}{d\theta}\right)^2\frac{du}{d\theta}\cos^3\theta\sin\theta}{\cos^2 i}$$

$$+ \left(\frac{di}{d\theta}\right)^2\frac{du}{d\theta}\cos^3\theta\sin\theta + \frac{4\sin^3 i\frac{di}{d\theta}\frac{d\Omega}{d\theta}u\cos^3\theta\sin\theta}{\cos^2 i}$$

$$+ \frac{2\sin i\frac{di}{d\theta}\frac{d^2\Omega}{d\theta^2}\frac{du}{d\theta}\cos\theta\sin\theta}{\cos^2 i}$$

$$+ \frac{6\sin^2 i\left(\frac{d\Omega}{d\theta}\right)^2\frac{du}{d\theta}\cos\theta\sin\theta}{\cos^2 i}$$

$$+ \frac{2\sin i\frac{d^2i}{d\theta^2}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\cos\theta\sin\theta}{\cos^2 i}$$

$$+ \sin i\frac{d^2i}{d\theta^2}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\cos\theta\sin\theta$$

$$- \frac{2\sin^2 i\frac{d\Omega}{d\theta}\frac{du}{d\theta}\cos\theta\sin\theta}{\cos i} - \frac{\sin i\frac{d^2i}{d\theta^2}\frac{du}{d\theta}\cos\theta\sin\theta}{\cos i}$$

$$+ \frac{2\sin^2 i\left(\frac{di}{d\theta}\right)^2\frac{du}{d\theta}\cos\theta\sin\theta}{\cos^2 i} - \left(\frac{di}{d\theta}\right)^2\frac{du}{d\theta}\cos\theta\sin\theta$$

$$+ 4\sin i\frac{di}{d\theta}\frac{d\Omega}{d\theta}u\cos\theta\sin\theta - \frac{2\sin i\frac{di}{d\theta}u\cos\theta\sin\theta}{\cos i}$$

$$+ \frac{\sin^3 i \frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\cos^4\theta}{\cos^2 i} - \frac{3\sin^4 i\left(\frac{d\Omega}{d\theta}\right)^2 u\cos^4\theta}{\cos^2 i}$$

$$+ \frac{\sin^3 i \frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\cos^2\theta}{\cos^2 i} + \frac{6\sin i \frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\cos^2\theta}{\cos^2 i}$$

$$+ 3\sin i\frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{du}{d\theta}\cos^2\theta - \frac{3\sin i\frac{di}{d\theta}\frac{du}{d\theta}\cos^2\theta}{\cos i}$$

$$- 3\sin^2 i\left(\frac{d\Omega}{d\theta}\right)^2 u\cos^2\theta + \frac{2\sin^2 i\frac{d\Omega}{d\theta}u\cos^2\theta}{\cos i} - \frac{3\left(\frac{d\Omega}{d\theta}\right)^2\frac{d^2u}{d\theta^2}}{\cos^2 i}$$

$$+ \frac{2\frac{d\Omega}{d\theta}\frac{d^2u}{d\theta^2}}{\cos i} - \frac{3\frac{d\Omega}{d\theta}\frac{d^2\Omega}{d\theta^2}\frac{du}{d\theta}}{\cos^2 i} + \frac{\frac{d^2\Omega}{d\theta^2}\frac{du}{d\theta}}{\cos i}$$

$$- \frac{2\sin i\frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{du}{d\theta}}{\cos^2 i} - \sin i\frac{di}{d\theta}\frac{d\Omega}{d\theta}\frac{du}{d\theta} + \frac{\sin i\frac{di}{d\theta}\frac{du}{d\theta}}{\cos i}$$

$$+ \frac{\cos^2 i J u^2}{\cos^2 i_0} + \frac{\cos^2 i}{\cos^2 i_0}. \tag{5.24}$$

Equations (5.23) and (5.24) can be greatly simplified if only a first order approximation to the equations of motion is desired. The approximate equations to (5.23) and (5.24) are respectively:

$$-2\sin i\frac{d\Omega}{d\theta}\sin\theta - \frac{d^2i}{d\theta^2}\sin\theta + \sin i\frac{d^2\Omega}{d\theta^2}\cos\theta - 2\frac{di}{d\theta}\cos\theta = \frac{2\cos^3 i\sin i\, J\, u\sin\theta}{\cos^2 i_0} \tag{5.25}$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{\sin i\frac{di}{d\theta}\frac{d^2u}{d\theta^2}\sin(2\theta)}{\cos i}$$

$$- \frac{2\sin^2 i\frac{d\Omega}{d\theta}\frac{d^2u}{d\theta^2}\sin^2\theta}{\cos i} - \frac{\sin^2 i\frac{d^2\Omega}{d\theta^2}\frac{du}{d\theta}\sin^2\theta}{\cos^2 i} - \frac{\sin i\frac{di}{d\theta}\frac{du}{d\theta}\sin^2\theta}{\cos i}$$

$$- \frac{3\cos^2 i\sin^2 i\, J u^2\sin^2\theta}{\cos^2 i} - \frac{2\sin^2 i\frac{d\Omega}{d\theta}\frac{du}{d\theta}\cos\theta\sin\theta}{\cos i_0}$$

$$- \frac{\sin i\frac{d^2 i}{d\theta^2}\frac{du}{d\theta}\cos\theta\sin\theta}{\cos i} - \frac{2\sin i\frac{di}{d\theta}u\cos\theta\sin\theta}{\cos i}$$

$$
\begin{aligned}
&- \frac{3\sin i \frac{di}{d\theta}\frac{du}{d\theta}\cos^2\theta}{\cos i} + \frac{2\sin^2 i \frac{d\Omega}{d\theta} u \cos^2\theta}{\cos i} \\
&+ \frac{2\frac{d\Omega}{d\theta}\frac{d^2u}{d\theta^2}}{\cos i} + \frac{\frac{d^2\Omega}{d\theta^2}\frac{du}{d\theta}}{\cos i} + \frac{\sin i \frac{di}{d\theta}\frac{du}{d\theta}}{\cos i} + \frac{\cos^2 i\, J\, u^2}{\cos^2 i_0} \\
&+ \frac{\cos^2 i}{\cos^2 i_0}. \qquad (5.26)
\end{aligned}
$$

Equations (5.25) and (5.26) will be used in the initial analysis; the general equations will be used for second order calculations.

F. THE INITIAL CONDITIONS

It is desired that the solution to the equations derived in the last section be an accurate, long term predictor of the satellite's motion. In fact, as long as the oblateness perturbation remains the dominant disturbing force, the solution should be valid for close to 1000 revolutions. However, before the solution can be used for prediction, a set of initial conditions must be determined.

The subject of orbit determination represents a separate discipline within celestial mechanics, and a discussion of the various techniques used to determine an orbit is outside the scope of this analysis. It is sufficient therefore, to state that the purpose of orbit determination is to find the orbital elements of a satellite from reduced observational data. A set of observations will determine the osculating elements at time t_0. As was noted in Chapter 4, these elements will change, and at t_1 a new set of osculating elements may be calculated. For the purpose of this analysis any observed set may be designated as the osculating elements at t_0 and thus prescribe the initial conditions. Stated mathematically, the initial conditions are at $t = t_0$:

$$
r = \frac{a(1-e^2)}{1 + e\cos(\theta_0 - \omega)} \qquad \frac{dr}{d\theta} = \frac{a(1-e^2)e\sin(\theta_0 - \omega)}{(1 + e\cos(\theta_0 - \omega))^2}
$$

where $a = a_0$, $e = e_0$, $\omega = \omega_0$,

$$\text{and } i = i_0 \qquad \Omega = \Omega_0 \qquad y = \theta_0 - \omega \tag{5.27}$$

VI. THE PERTURBATION PROCEDURE

A. PRELIMINARIES

The perturbation method used in this analysis is a variation of tne technique known as the Method of Strained Coordinates. The motivation for the use of this technique is the subject of this section.

In perturbation theory, the quantities to be expanded can be functions of one or more variables besides the perturbation coordinate. An asymptotic expansion of $f(\theta; J)$ in terms of the asymptotic sequence $\epsilon_m(J)$ is,

$$f(\theta; J) \sim \sum_{m=0}^{\infty} a_m(\theta)\epsilon_m(J) \qquad \text{as} \quad J \to 0 \tag{6.1}$$

where θ is a scalar (or vector) variable independent of J and the coefficients a_m are functions of θ only. This expansion is said to be uniformly valid if,

$$\begin{aligned} f(\theta; J) &= \sum_{m=0}^{n-1} a_m(\theta)\epsilon_m(J) + R_n(\theta; J) \\ R_n(\theta; J) &= O(\epsilon_n(J)). \end{aligned}$$

For these uniformity conditions to hold, $a_m(\theta)\epsilon_m(J)$ must be small compared to the preceding term $a_{m-1}(\theta)\epsilon_{m-1}(J)$ for each m. Each term must be a small correction to the preceding term regardless of the value of θ.

Unfortunately, it is the rule rather than the exception that expansions like (6.1) are non-uniformly valid and break down in certain cases. A case of particular interest is the presence of secular terms such as $\theta^n \cos\theta$ and $\theta^n \sin\theta$ which make $f_m(\theta)/f_{m-1}(\theta)$ unbounded as θ approaches infinity.

To illustrate how secular terms arise in the solutions to differential equations, reference is made to one of the equations of motion derived in the preceding

chapter. Equation (5.26) has the form,

$$u'' + u = 1 + J\, f(u, u', u'', i', i'', \Omega', \Omega'')\,.$$

When expanded, the right side will be linear combinations of trigonometric terms and constants. The presence of terms of the form $\cos(\theta)$, $\sin(\theta)$, $\cos(3y - 2\theta)$, $\sin(2y - \theta)$, etc., on the right side will produce secular terms. For example, the second order differential equation,

$$u'' + u = \cos\ \theta$$

has the solution,

$$u = \frac{\theta \sin\ \theta + \cos\ \theta}{2} + K_1 \sin\ \theta + K_2 \cos\ \theta.$$

Note that the presence of the term $\theta \sin\ \theta$ will result in unbounded solutions for u as θ approaches infinity. Since in this representation u is the reciprocal of the distance from the center of the planet to the satellite, the physical effect of the secular terms would be to produce in r *periodic terms* with large amplitude variations, a situation that certainly does not correspond to physical reality.

A technique for dealing with terms that produce secular terms is to eliminate them from the right side of the differential equation. The Method of Strained Coordinates is a perturbation technique designed for removing secular terms. To illustrate, it is recalled that the solution to the two body problem is

$$1/r = u = 1 + e\ \cos(\theta - \omega), \qquad \text{where} \quad p = 1.$$

In the Method of Strained Coordinates, the θ coordinate is strained by introducing a new variable $y = f\,(\theta) = \theta - \omega + J y_1 \theta + \ldots$ As will be shown in the next section, choosing the correct value for y_1 will insure secular terms do not arise in the first order solution for u.

While the above method removes secular terms to first order, it is not adequate for dealing with secular terms that arise to second order in the equations of motion. That it is important to remove second order secular terms is shown by the following equation:

$$
\begin{aligned}
f(r) &= 1 + e\cos(\theta - \omega) + J(\cos\theta + \ldots) \\
&+ J^2(\theta\cos(\theta - \omega) + \ldots) + J^3(\theta^2\cos(\theta - \omega) + \ldots) + \ldots
\end{aligned}
$$

Note that, although the terms through order J are bounded, as $\theta \to \infty$, the J^2 and J^3 terms grow without bound and dominate the right side. In this present analysis, θ has an upper bound of $(2\pi)10^3$; however, the effect of secular terms remains since $J^2\theta, J^3\theta^2$, etc., are all of order J as $\theta \to (2\pi)10^3$. An infinite series would have to be retained.

In this present work, three additional techniques had to be devised to deal with secular terms to second order. These techniques will be discussed in the next section. The perturbation method therefore is not strictly the Method of Strained Coordinates, but a variation of that method.

The basic steps in the perturbation procedure are as follows:

1. The dependent variables and independent variables are expanded as functions of a small parameter (J).
2. The variables are then substituted into the equations of motion, and the equations are solved consecutively. Each solution yields a more exact expression for the appropriate variable.

The process is carried out through second order to demonstrate that all secular terms may be eliminated and that the solutions are bounded. The following section highlights the calculations involved in the process and shows the first order equations and solutions. The second order expressions are long, and their display

in this context would not contribute to the analysis since only a few specific terms are relevant. The complete expressions are contained in Appendix B; they will be referred to during the course of the analysis.

C. SOLVING THE EQUATIONS

1. First Order Approximation for i and Ω

The equations to be solved are (5.25) and (5.26). Since the right side of these equations are analytic functions of J, it is reasonable to expect that the solution to u will be arbitrarily close to the two-body solution, $1 + e\cos(\theta - \omega)$, when J is sufficiently close to 0. Likewise i, Ω, and y will be arbitrarily close to i_0, Ω_0, and $\theta - \omega$, respectively, when J is close to 0. This assumption amounts to letting

$$u = 1 + e\cos y + Ju_1 + J^2u_2 + \ldots \tag{5.2}$$

$$y = \theta - \omega + Jy_1 + J^2y_2 + \ldots \tag{5.3}$$

$$i = i_0 + Ji_1 + J^2i_2 + \ldots \tag{5.4}$$

$$\Omega = \Omega_0 + J\Omega_1 + J^2\Omega_2 + \ldots \tag{5.5}$$

The first step in the solution of the equations of motion is to substitute (6.2) - (6.5) into equation (5.25) and equate the coefficients of J. The result is

$$\begin{aligned}&(-2\Omega_1'\sin\theta + \Omega_1''\cos\theta)\sin i_0 - 2i_1'\cos\theta - i_1''\sin\theta \\ &= 2\cos i_0\sin i_0\sin\theta(1 + e\cos y)\,.\end{aligned} \tag{5.6}$$

Sin i and cos i have been replaced in the above equations by their approximations: sin i_0 and cos i_0. These are valid approximations since Ω'', Ω', i'', and i' are all of order J.

Equation (6.6) is a linear differential equation with two unknown functions. Terms of the form cos θ, sin θ, etc., on the right side of this equation, and the more general equation (5.23), will produce secular terms in i and Ω. It should be recalled that these same terms cause secular terms in u. Note that (6.6) contains a sin θ terms on the right side. There would be a need to eliminate this term were it not for the conditions placed on Ω and i by the definition of the reference plane. Specifically, Ω, which governs the rotation of the reference plane, must be expressible as an unbounded (secular) term plus bounded periodic terms, and i must be bounded. The fact that Ω must contain a secular term negates the requirement to eliminate the sin θ term. Therefore, a solution of the form

$$\begin{aligned} \Omega_1 &= \alpha_1\theta + \alpha_2 \sin y \\ i_1 &= \beta_1 \cos y \end{aligned} \tag{5.7}$$

is assumed.

By substituting (6.7) into (6.6) and equating coefficients, the following solution for i and Ω is obtained:

$$\begin{aligned} i_1 &= -2/3\, e \cos i_0 \sin i_0 \cos y \\ \Omega_1 &= -\theta \cos i_0 - 4/3\, e \cos i_0 \sin y \end{aligned} \tag{5.8}$$

2. Second Order Secular Terms in i and Ω

Equation (6.8) satisfies (6.6) to order J. The next step in the process is to substitute (6.2) - (6.5) and (6.8) into (5.26) and solve for u_1. However, proceeding with (6.8) in its present form will lead to secular terms in second order. A brief paragraph will explain why secular terms are anticipated.

Success in using perturbation methods requires a certain *a priori* knowledge about the nature of the particular problem one is trying to solve. There is a

certain amount of trial and error involved in the process. For instance, the success of the Method of Strained Coordinates depends on the knowledge that a secular term will arise in the first order solution to (5.26) and that a mechanism (the strained coordinate) can be put in place one step ahead in order to eliminate the secular term. The second order secular terms were found by this same trial and error process.

Returning now to equation (6.8), it was discovered in this analysis that the problem terms

$$\frac{(-45s^3 + 28s)}{24} J^2 e^2 c \sin(2y - \theta) \tag{6.9}$$

and

$$\frac{5}{8} J^2 s^3 e^2 c \sin(2y - 3\theta)$$

$$(\text{where } s = \sin \; i_0, \quad c = \cos i_0).$$

appear on the right side of (5.23) when it is expressed to order J^2. The appearance of these terms that give rise to secular terms was unexpected. They were not reported by Brenner because the authors assumed a small eccentricity and dropped $J^2 e^2$ and higher terms.

There were several failed attempts in dealing with these new terms before an effective measure was discovered. First, an investigation was made into the effect of retaining the secular terms.

The secular terms produced by (6.9) are:

$$\begin{aligned} i_2 &= \frac{(15s^3 - 14s)}{24} e^2 c \, \theta \sin(2y - 2\theta) \\ \Omega_2 &= \frac{(15s^2 - 7)}{12} e^2 c \, \theta \cos(2y - 2\theta) \, . \end{aligned}$$

As was demonstrated earlier, when $\theta \to (2\pi)10^3$ these terms are of order J and must be retained in the first order solution. As the solution progresses, these terms

will continue to produce secular terms with coefficients $J^3\theta^2$, $J^4\theta^3$, ..., $J^n\theta^{n-1}$. A convergent series representation for these terms was not found.

The remaining alternative was to alter the form of (6.7) to eliminate secular terms. If terms with arbitrary coefficients can be found which when added to (6.7) produce terms of identical harmonics to the same order as (6.9), the coefficients may be chosen so that all terms with these harmonics are eliminated. Essentially, the only terms that may be added to (6.7) are terms which satisfy the original differential equation, equation (5.23). Therefore, one may add homogeneous solutions of the differential equation or arbitrary constants.

Adding an arbitrary constant to (6.7) has the following effect on higher order terms. From equations (5.23) and (5.24), it would appear that only the derivatives of i and Ω enter into the higher order calculations, and therefore arbitrary constants would be eliminated. This is true for Ω, but not for i. To explain, it is recalled that the approximations $\sin i = \sin i_0$ and $\cos i = \cos i_0$ were valid for the first order approximation of i_1 and Ω_1. This approximation was valid since all terms were of order J. However, in the calculation for u_1 there is a term $\cos^2 i/\cos^2 i_0$ that is of order 1 (Eq. (5.26)), therefore a better approximation for i is required. Using (6.7) and the Taylor series expansion for $\cos i$ (keeping only terms of order J) results in

$$
\begin{aligned}
\cos i &= \cos(i_0 + f(J)) = \cos i_0 - \sin i_0 f(J) \\
&= \cos i_0 - \sin i_0(i_1)
\end{aligned}
$$

Adding an arbitrary constant K_α to (6.8) results in

$$\cos i = \cos i_0 + \sin i_0(2/3\, J\, e \cos i_0 \sin i_0 \cos y + K_\alpha)$$

(A similar expression is required for $\sin i$)

Therefore, a constant added to (6.8) will alter the form of subsequent terms, but not in the form required to eliminate secular terms in i_2 and Ω_2. It will be shown that a constant like K_α will produce a term of the form $K_\alpha J \sin^2 i_0$ in u_1 and terms of the form $K_\alpha J^2 e \sin^2 i_0 \cos i_0 \cos y$ in i_2, and $K_\alpha J^2 e \sin^2 i_0 \cos i_0 \sin y$ in Ω_2. Although they cannot be used to eliminate secular terms, the constants will be needed to satisfy the initial condition, therefore it is essential that they do not produce irremovable secular terms to higher order.

The next alternative for the elimination of secular terms is to add a solution to the homogeneous equation of (5.23). While it will be shown that this technique is successful in eliminating secular terms in u_2, it did not succeed in removing them in i_2 and Ω_2. It failed because the homogeneous terms produced new secular terms to higher order. Many various combinations of homogeneous solutions were added to (6.7), but all attempts along this line only complicated the problem.

An answer to the question of how to eliminate the secular terms was suggested in a report by Weisfield [Ref. 31] on polar orbits. When faced with the problem of eliminating secular terms from his equation for Δ^2, Weisfield added a term like $\cos(2y - 2\theta)$. Now to the particular order to which one is working this term acts like a constant in the derivative, e.g., let,

$$y = \theta + J\theta$$

then

$$\frac{d}{d\theta}(\cos(2y - 2\theta)) = \sin(2y - 2\theta)(2 - 2(1 + J)) = O(J).$$

The attempt was then made to apply Weisfield's technique to this more general problem. (6.7) would then become

$$
\begin{aligned}
i_1 &= -2/3e \cos i_0 \sin i_0 \cos y + \bar{K}_1 \cos(2y - 2\theta) + \bar{K}_2 \\
\Omega_1 &= \theta \cos i_0 - 4/3e \cos i_0 \sin y + \bar{K}_3 \sin(2y - 2\theta) + \bar{K}_4 . \qquad (6.10)
\end{aligned}
$$

where: $\bar{K}_1 = -K_1 e^2 cs \quad \bar{K}_2 = K_1 \cos(2\omega) + K_2$
$\bar{K}_3 = -K_3 e^2 c \quad \bar{K}_4 = -K_3 \sin(2\omega) + K_4$

(It is noted that (6.10) still satisfies (6.6) to order J)

The added harmonic terms produce identical harmonics as (6.9) to second order and no other new secular terms. Thus, they allow for the elimination of these problem terms.

3. First Order Approximation for u

With valid expressions for i_1 and Ω_1 established, the procedure may now be continued with the calculation of u_1. Substitution of (6.2) - (6.5) and (6.10) into (5.26) and letting $\sin i_0 = s$ and $\cos i_0 = c$ results in the following equation

$$
\begin{aligned}
\frac{d^2 u_1}{d\theta^2} + u_1 &= 2e \cos y\, y_1 + \frac{5e^2 s^2 \cos(2y + 2\theta)}{8} \\
&+ 2e^2 K_1 s^2 \cos(2y - 2\theta) + \frac{e^2 s^2 \cos(2y - 2\theta)}{8} \\
&+ \frac{5es^2 \cos(y + 2\theta)}{3} - \frac{11e^2 s^2 \cos(2y)}{4} + \frac{5e^2 \cos(2y)}{2} \\
&- 5es^2 \cos y + 4e \cos y + \frac{3e^2 s^2 \cos(2\theta)}{4} + \frac{s^2 \cos(2\theta)}{2} \\
&- 2e^2 K_1 s^2 \cos(\omega) - 2K_2 s^2 - \frac{17e^2 s^2}{12} - \frac{5s^2}{2} + \frac{7e^2}{6} + 1 . \qquad (6.11)
\end{aligned}
$$

In the above equation, the cos y terms will produce secular terms in u_1. A choice of $y_1 = (5s^2 - 4)/2$ will eliminate that possibility. y becomes

$$
y = \theta - \omega - J\,\theta\left(\frac{5}{2}s^2 - 2\right) + J^2 y_2 \theta .
$$

Equation (6.11) becomes

$$
\begin{aligned}
\frac{d^2u_1}{d\theta^2}+u_1 &= \frac{5e^2s^2(2y+2\theta)}{8}+2e^2K_1s^2\cos(2y-2\theta)\\
&+\frac{e^2s^2\cos(2y-2\theta)}{8}+\frac{5es^2\cos(y+2\theta)}{3}-\frac{11e^2s^2\cos(2y)}{4}\\
&+\frac{5e^2\cos(2y)}{2}+\frac{3e^2s^2\cos(2\theta)}{4}+\frac{s^2\cos(2\theta)}{2}\\
&-2e^2K_1s^2\cos(2\omega)-2K_2s^2-\frac{17e^2s^2}{12}-\frac{5s^2}{2}+\frac{7e^2}{6}+1. \qquad (6.12)
\end{aligned}
$$

To solve (6.12), a solution of the form

$$
\begin{aligned}
u_1 &= \alpha_1\cos(2y+2\theta)+\alpha_2\cos(2y-2\theta)+\alpha_3\cos(y+2\theta)+\alpha_4\cos(2y)\\
&\quad \alpha_5\cos(2\theta)+\alpha_0 \qquad (6.13)
\end{aligned}
$$

is assumed.

Equation (6.13) is substituted into (6.12) and the coefficients of like harmonics are equated. The coefficients are:

$$
\begin{aligned}
\alpha_1 &= \frac{-e^2s^2}{24}\\
\alpha_2 &= \left(2K_1+\frac{1}{8}\right)e^2s^2\\
\alpha_3 &= \frac{-5es^2}{24}\\
\alpha_4 &= \left(\frac{11s^2}{12}-\frac{5}{6}\right)e^2\\
\alpha_5 &= -\left(\frac{e}{4}+\frac{1}{6}\right)\\
\alpha_0 &= -2K_1e^2s^2\cos(2\omega)-2K_2s^2-\left(\frac{17e^2}{12}+\frac{5}{2}\right)s^2\\
&\quad +\frac{7}{6}e^2+1
\end{aligned}
$$

Equating (6.13) for u_1 results in:

$$\begin{aligned}
u_1 &= -\frac{e^2s^2\cos(2y+2\theta)}{24} + 2e^2K_1s^2\cos(2y-2\theta) \\
&+ \frac{e^2s^2\cos(2y-2\theta)}{8} - \frac{5es^2\cos(y+2\theta)}{24} + \frac{11e^2s^2\cos(2y)}{12} \\
&- \frac{5e^2\cos(2y)}{6} - \frac{e^2s^2\cos(2\theta)}{4} - \frac{s^2\cos(2\theta)}{6} - 2e^2K_1s^2\cos(2\omega) \\
&- 2K_2s^2 - \frac{17e^2s^2}{12} - \frac{5s^2}{2} + \frac{7e^2}{6} + 1. \qquad (6.14)
\end{aligned}$$

4. Second Order Secular Terms in u

The above expression for u_1 is not complete. Looking one step ahead it is known (by trial and error) that the following problem terms arise in the differential equation for u_2.

$$\begin{aligned}
\frac{d^2u_2}{d\theta^2} + u_2 &= \left(\frac{75e^3s^4 - 200e^3s^2 + 136e^3}{240} - \frac{e^3}{15(5s^2-4)}\right)\cos(3y-2\theta) \\
&+ \left(\frac{375e^3s^4 + (-480e^3 - 40e)s^2 + 136e^3}{240} - \frac{e^3}{15(5s^2-4)}\right)\cos(y-2\theta) \\
&+ \left(2ey_2 + \frac{(225e^3s^4 - 225e^3s^2 + 2e^3)\cos(2\omega)}{60} + \frac{2e^3\cos(2\omega)}{75s^2-60}\right. \\
&- \left.\frac{(45e^3 - 550e)s^4 + (36e^3 + 488e)s^2 - 56e^3}{48}\right)\cos(y) \qquad (6.15)
\end{aligned}$$

(Only the problem terms are displayed. The complete expression is equation (B.5), App. B.)

The harmonics $\cos(3y-2\theta)$, $\cos(y-2\theta)$, and $\cos y$ all cause secular terms in u_2, and therefore must be eliminated. In addition there is another more troubling problem with (6.15). That is, certain terms have in their denominator $(5\sin^2 i_0 - 4)$, and if $i_0 = \pm\sin^{-1}\sqrt{4/5}$, then the denominators are zero. This in-

clination is well known in Astrodynamics, and has been named the critical inclination, or more appropriately the critical inclinations since there are two: $i_0 = 63^0 26'$ and $117^0 34'$. The problem of the critical inclination will be dealt with later in the analysis. In addition, there is qualitative discussion of the critical inclination in Appendix C.

The task is to eliminate the three terms which give rise to secular terms in u_2 from the right side of (6.15). By inspection it is seen that a proper choice of y_2 will eliminate secular terms produced by cos y. It was then discovered in this analysis that the addition of a term of the form $\cos(y-2\theta)-\cos(y+2\omega)$ eliminated the $\cos(y-2\theta)$ problem term. This term may be added since it is a solution to the homogeneous equation. There remained one term to eliminate; $\cos(3y-2\theta)$. This term can be eliminated by adding a term with the harmonic $\sin(2y-2\theta)$ to y_1. In addition, a constant was added to y_1 to satisfy the initial conditions. The complete first order expression for y is now

$$\begin{aligned} y &= \theta - \omega + J\left(\theta\left(\frac{5}{2}s^2 - 2\right) + \bar{K}_5 \sin(2y-2\theta) + \bar{K}_6\right) \\ &\quad + J^2 y_2 \theta . \end{aligned} \tag{6.16}$$

where $\bar{K}_5 = -JK_5$ and $\bar{K}_6 = -JK_5\sin(2\omega) + K_6$.

By adding $-K_7(\cos(y-2\theta)+\cos(y+2\omega))$ to eliminate a secular term and K_8 cos y $+K_9 \sin y$ to satisfy initial conditions, u_1 becomes

$$\begin{aligned} u_1 &= -\frac{e^2 s^2 \cos(2y+2\theta)}{24} + 2e^2 K_1 s^2 \cos(2y-2\theta) \\ &+ \frac{e^2 s^2 \cos(2y-2\theta)}{8} - \frac{5es^2\cos(y+2\theta)}{24} - eK_7\cos(y-2\theta) \\ &+ eK_7\cos(y+2\omega) + \frac{11e^2s^2\cos(2y)}{12} - \frac{5e^2\cos(2y)}{6} + K_9\sin(y) \\ &+ K_8\cos(y) - \frac{e^2s^2\cos(2\theta)}{4} - \frac{s^2\cos(2\theta)}{6} - 2e^2K_1s^2\cos(2\omega) \end{aligned}$$

$$- \quad 2K_2 s^2 - \frac{17e^2s^2}{12} - \frac{5s^2}{2} + \frac{7e^2}{6} + 1 . \tag{6.17}$$

5. Second Order Solution for i and Ω

With all terms in place to deal with secular terms the calculations are continued by substituting (6.2) - (6.5), (6.10), (6.16), and (6.17) into (5.23) to solve for Ω_2 and i_2, and to evaluate the constants K_1 and K_3. The result is contained in Appendix B, i.e.,

$$\begin{aligned}
&(-2\Omega_2' \sin\theta + \Omega_2'' \cos\theta) \sin i_0 - 2i_2' \cos\theta - i_2'' \sin\theta \\
&= \text{ Right side (B.1), (Appendix B).}
\end{aligned}$$

An inspection of (B.1) reveals that the coefficients of $\sin(2y - \theta)$ and $\sin(2y - 3\theta)$ form respectively the following simultaneous equations:

$$\frac{(5K_3 + 10K_1 - 45)e^2s^3c}{24} - \frac{(4K_3 + 4K_1 - 7)e^2s\,c}{6} = 0$$

$$\left(\frac{5}{8} - 5K_3\right) e^2s^3c + (4K_1 - 4K_3)e^2s\,c = 0.$$

Solving these equations results in

$$K_1 = \frac{15s^2 - 14}{24(5s^2 - 4)} \tag{6.18}$$

$$K_3 = \frac{75s^4 - 120s^2 + 56}{24(5s^2 - 4)^2}. \tag{6.19}$$

Substitution of these values for K_1 and K_3 into (B.1) gets rid of all $\sin(y - 2\theta)$ and $\sin(2y - 3\theta)$ terms. With an assurance that no secular terms will arise, the equation for Ω_2 and i_2 can be solved.

Before progressing, it is noted that (6.18) and (6.19) contain the same problem denominator that was observed in (6.15). In fact, (6.18) and (6.19) are the first occurrence of the critical inclination term in this analysis. The term then

continues to manifest itself in all higher order analyses. The reason for the presence of the term is simple to explain. The constants K_1 and K_3 were multiplied by the derivative of y_1 $\left(\frac{dy_1}{d\theta} = \left(\frac{5}{2}s^2 - 2\right)\right)$ during the analysis. The derivative then shows up in the denominator of these constants when they are evaluated. It will be important to show later in the results that despite the occurrence of an apparent singularity, the final solution is uniformly valid for all inclinations, including the critical.

With secular terms removed the following equation can be solved for Ω_2 and i_2.

$$(-2\Omega_2' \sin\theta + \Omega_2'' \cos\theta)\sin i_0 - 2i_2' \cos\theta - i_2'' \sin\theta \tag{6.20}$$

$$= \text{Right side (B.2), (Appendix B.).}$$

As was done in (6.6), a solution which includes the harmonics of the right side of (B.2) is assumed for Ω_2 and i_2. This solution is substituted into (B.2), and the coefficients of the various harmonics are equated. Once again the conditions are that Ω_2 must be expressed as a secular term and bounded periodic terms, while i_2 must be bounded. The solution is

$$\begin{aligned} \Omega_2 &= \text{Right side (B.3)} \\ i_2 &= \text{Right side (B.4).} \end{aligned} \tag{6.21}$$

6. Eliminating the Final Secular Terms

To complete the solution the remaining constants must be found. Three of the remaining six constants, y_2, K_5 and K_7, are obtained from the differential equation for u_2. It is recalled that these constants are used to eliminate secular terms in u_2. There is no need to solve the equation for u_2 since the constants may be evaluated from the right side of the differential equation. Therefore, (6.2) -

(6.5), (6.10), (6.16), (6.17), and (6.21) are substituted into (5.24). The resulting equation is (B.5).

As in the previous solution for Ω_2 and i_2, the secular terms in u_2 are eliminated by setting the coefficients of the problem terms, i.e., (6.17), equal to zero and solving for the constants. The results are:

$$\begin{aligned}
K_5 &= \frac{75e^2s^6 - 260e^2s^4 + 296e^2s^2 - 112e^2}{192\left(\frac{5}{2}s^2 - 2\right)^2} \\
K_7 &= -\frac{15e^2s^4 - (14e^2 - 2)s^2}{48\left(\frac{5}{2}s^2 - 2\right)} \\
y_2 &= -\frac{e^2\cos(2\omega)}{30\left(\frac{5}{2}s^2 - 2\right)} + \bar{y}_2 \qquad (6.22)
\end{aligned}$$

where

$$\begin{aligned}
\bar{y}_2 &= -10es^4\cos(\omega - \theta_0) + \frac{26es^2\cos(\omega - \theta_0)}{3} - \frac{15e^2s^4\cos(2\omega)}{8} \\
&+ \frac{15e^2s^2\cos(2\omega)}{8} - \frac{e^2\cos(2\omega)}{60} + \frac{15e^2s^4}{32} - \frac{275s^4}{48} + \frac{3e^2s^2}{8} + \frac{61s^2}{12} \\
&- \frac{7e^2}{12}
\end{aligned}$$

With the above constants evaluated, it is assured that the second order expression is free of secular terms.

7. The Initial Conditions for y, i, θ, Ω

The task now is to evaluate the remaining constants by establishing the initial conditions. At $t = t_0$, it is required that the velocity vector of the satellite in the reference plane be tangent to the corresponding two body ellipse determined by the satellite. In addition to the initial conditions established for r and $\frac{dr}{d\theta}$ in (5.27), it is recalled that at $t = t_0$

$$y = \theta_0 - \omega, \qquad i = i_0 \qquad \text{and} \qquad \Omega = \Omega_0$$

From (6.16) and (6.22):

$$
\begin{aligned}
y &= \theta - \omega + J\left(\theta\left(\frac{5}{2}s^2 - 2\right)\right. \\
&- \left.\frac{(75e^2s^6 - 260e^2s^4 + 296e^2s^2 - 112e^2)}{192\left(\frac{5}{2}s^2 - 2\right)^2}(\sin(2y - 2\theta) - \sin(2\omega))\right) \\
&+ K_6 - \frac{e^2J^2\theta\cos(2\omega)}{30\left(\frac{5}{2}s^2 - 2\right)} + J^2\theta\,\bar{y}_2.
\end{aligned}
\tag{6.23}
$$

Choose K_6 such that at $t = t_0$, $y = \theta_0 - \omega$; therefore,

$$
K_6 = -J\theta_0\left(\frac{5}{2}s^2 - 2\right). \tag{6.24}
$$

To obtain the initial condition for i, use (6.10) and (6.18) so that

$$
i = i_0 - 2/3Jecs\cos y - Je^2cs\frac{(15s^2 - 14)(\cos(2y - 2\theta) - \cos(2\omega))}{24(5s^2 - 4)} + K_2. \tag{6.25}
$$

It is evident that the Je^2 terms are eliminated when $y = \theta - \omega$ and $\theta = \theta_0$. The result

$$
K_2 = 2/3Jecs\cos(\theta_0 - \omega) \tag{6.26}
$$

gives the desired $i = i_0$ at $t = t_0$.

For Ω, (6.10) and (6.19) are required. In addition all secular terms from Ω_2, equation (B.3), App. B, are needed since all these terms are of order J. Therefore,

$$
\begin{aligned}
\Omega &= \Omega_0 - J\theta c + \frac{ce^2J(75s^4 - 102s^2 + 56)(\sin(2y - 2\theta) + \sin(2\omega))}{24(5s^2 - 4)^2} \\
&- \frac{4ceJ\sin y}{3} - \frac{ce^2J^2\cos(2\omega)\theta}{15s^2 - 12} + \frac{5ce^2J^2s^2\cos(2\omega)\theta}{8} \\
&- \frac{ce^2J^2\cos(2\omega)\theta}{12} + 5cJ^2K_2s^2\theta - \frac{5ce^2J^2s^2\theta}{24} + \frac{5cJ^2s^2\theta}{3} \\
&- \frac{ce^2J^2\theta}{6} - \frac{cJ^2\theta}{2} + K_4
\end{aligned}
\tag{6.27}
$$

and

$$K_4 = Jc\theta_0 + \frac{4}{3}Jce\sin(\theta_0 - \omega). \tag{6.28}$$

8. θ As a Function of Time. (Initial Conditions for r, $\frac{dr}{d\theta}$)

To complete the evaluation of the remaining constants in u, it is necessary to give θ as a function of time. The expression for the perturbed $\frac{dt}{d\theta}$ then can be related to the well-known two-body formula, and from this relationship, an expression for r and $\frac{dr}{d\theta}$ may be derived. The task will be to choose K_8 and K_9 so the conditions as given by (5.27) are satisfied, i.e.,

$$r(t_0) = \frac{a(1-e^2)}{1+e\cos(\theta_0-\omega)} \qquad \frac{dr}{d\theta}(t_0) = \frac{a(1-e^2)e\sin(\theta_0-\omega)}{(1+e\cos(\theta_0-\omega))^2}$$

Proceeding in this manner, the formula for $\frac{dt}{d\theta}$ can be obtained from the relation

$$\frac{dt}{d\theta} = \frac{d\phi}{d\theta}\frac{dt}{d\phi} = \frac{d\phi}{d\theta}\frac{r^2\cos^2\delta}{h\cos i_0}$$

Direct substitution of the derived expression for $\frac{d\phi}{d\theta}$ equation (5.21), results in

$$\begin{aligned}
\frac{dt}{d\theta} &= \frac{r^2}{h}\left[1 + J\left(-\frac{e^2\cos(2y-2\theta) - e^2\cos(2\omega)}{15(5s^2-4)} + \frac{e^2s^2\cos(2y-2\theta)}{8}\right.\right. \\
&- \frac{e^2\cos(2y-2\theta)}{60} - \frac{es^2\cos(y+2\theta)}{6} - \frac{es^2\cos(y-2\theta)}{2} \\
&+ \frac{4es^2\cos y}{3} - \frac{4e\cos y}{3} - \frac{s^2\cos(2\theta)}{2} - \frac{2es^2\cos(\omega-\theta_0)}{3} \\
&- \left.\left.\frac{e^2s^2\cos(2\omega)}{8} + \frac{e^2\cos(2\omega)}{60} + \frac{s^2}{2} - 1\right)\right]
\end{aligned} \tag{6.29}$$

At $t = t_0$

$$\frac{dt}{d\theta}(\theta_0) = \frac{r^2}{h}(\theta_0).$$

The above condition is an integral from the two-body problem. If (6.29) is evaluated at $\theta = \theta_0$, then $\frac{dt}{d\theta}$ may be expressed as

$$\frac{r^2}{\bar{h}}(1 + JK_{10}) = \frac{r^2}{h}$$

where

$$\begin{aligned} K_{10} &= -\frac{es^2\cos(\omega+\theta_0)}{2} + \frac{2es^2\cos(\omega-\theta_0)}{3} - \frac{4e\cos(\omega-\theta_0)}{2} \\ &- \frac{es^2\cos(\omega-3\theta_0)}{6} - \frac{s^2\cos(2\theta_0)}{2} + \frac{s^2}{2} - 1 \end{aligned}$$

From this expression, a formula for $\bar{h}$ results:

$$\bar{h} = h(1 + JK_{10})$$

so that from (6.29)

$$\frac{dt}{d\theta} = \frac{r^2}{h}(1 + J(-K_{10} + \ldots))$$

Using

$$\bar{p} = \frac{\bar{h}^2}{GM} = \frac{h^2}{GM}(1 + 2JK_{10}) = a(1-e^2)(1 + 2JK_{10})$$

a new formula for r, which includes the first order perturbation effects, may be written as

$$r = \frac{\bar{p}}{u} = \frac{a(1-e^2)}{1 + e\cos y + J(-2K_{10} + u_1)} \qquad (6.30)$$

where u_1 is (6.17).

The formula for $\frac{dr}{d\theta}$ is

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{a(1-e^2)e\sin y}{(1+e\cos y)^2} \\ &+ \frac{Ja(e^2-1)}{(1+e\cos y)^2}\left[\frac{e^2s^2\sin(2y+2\theta)}{6} + \frac{5es^2\sin(y+2\theta)}{8} - eK_7\sin(y-2\theta)\right. \end{aligned}$$

$$
\begin{aligned}
&- \quad eK_7 \sin(y+2\omega) - \frac{11e^2s^2\sin(2y)}{6} + \frac{5e^2\sin(2y)}{3} - \frac{5es^2\sin(y)}{2} \\
&- \quad K_8 \sin\, y + 2e \sin\, y + K_9 \cos\, y + \frac{e^2s^2\sin(2\theta)}{2} + \frac{s^2\sin(2\theta)}{3}\Bigg]
\end{aligned} \tag{6.31}
$$

Next, evaluate (6.30) and (6.31) at $\theta = \theta_0$. Keep only the terms to order J. The result is two simultaneous equations that can be solved for K_8 and K_9:

$$
\begin{aligned}
K_8 \quad = \quad & -\frac{e^2s^2\cos(3\omega-\theta_0)}{16} + \frac{11e^2s^2\cos(3\omega-3\theta_0)}{24} \\
& - \frac{5e^2\cos(3\omega-3\theta_0)}{12} - \frac{e^2s^2\cos(3\omega-5\theta_0)}{16} + \frac{31es^2\cos(2\omega-2\theta_0)}{12} \\
& - \frac{7e\cos(2\omega-2\theta_0)}{3} - \frac{3es^3\cos(2\omega-4\theta_0)}{8} - \frac{11e^2s^2\cos(\omega+\theta_0)}{16} \\
& - \frac{s^2\cos(\omega+\theta_0)}{4} + \frac{11e^2s^2\cos(\omega-\theta_0)}{8} + \frac{7s^2\cos(\omega-\theta_0)}{2} \\
& - \frac{31e^2\cos(\omega-\theta_0)}{12} - 3\cos(\omega-\theta_0) - \frac{17e^2s^2\cos(\omega-3\theta_0)}{48} \\
& - \frac{7s^2\cos(\omega-3\theta_0)}{12} - \frac{es^2\cos(2\omega)}{2} - \frac{5es^2\cos(2\theta_0)}{4} + \frac{13es^2}{12} - \frac{7e}{3}
\end{aligned} \tag{6.32}
$$

$$
\begin{aligned}
K_9 \quad = \quad & \frac{e^2s^2\sin(3\omega-\theta_0)}{16} - \frac{11e^2s^2\sin(3\omega-3\theta_0)}{24} + \frac{5e^2(3\omega-3\theta_0)}{12} \\
& - \frac{e^2s^2\sin(3\omega-5\theta_0)}{16} - \frac{31es^2\sin(2\omega-2\theta_0)}{12} + \frac{7e\sin(2\omega-2\theta_0)}{3} \\
& + \frac{3es^2\sin(2\omega-4\theta_0)}{8} - \frac{7e^2s^2\sin(\omega+\theta_0)}{16} + \frac{s^2\sin(\omega+\theta_0)}{4} \\
& - \frac{67e^2s^2\sin(\omega-\theta_0)}{24} - \frac{7s^2\sin(\omega-\theta_0)}{2} + \frac{29e^2\sin(\omega-\theta_0)}{12} \\
& - 3\sin(\omega-\theta_0) + \frac{11e^2s^2\sin(\omega-3\theta_0)}{48} + \frac{7s^2\sin(\omega-3\theta_0)}{12} \\
& - \frac{es^2\sin(2\omega)}{2} - \frac{3es^2\sin(2\theta_0)}{4}\,.
\end{aligned} \tag{6.33}
$$

Substitution of the values of K_8 and K_9 in the equation for u insures that the remaining initial conditions for r and $\frac{dr}{d\theta}$ are satisfied.

C. THE RESULTS

1. Preliminaries

It was noted that throughout the calculations in this chapter the term $\frac{(5\sin^2 i_0 - 2)}{2}$ appeared in the denominator of terms in u_1, i_2, Ω_2, and u_2. It would appear that the solution is not valid when $i_0 = \pm \sin^{-1}\sqrt{4/5}$. And if the solution is not valid near these "critical inclinations", then one should question the validity of the underlying perturbation process.

The purpose of this last section is to display the final solutions for y, i, Ω, and u, and to show that each of the apparent singularities can be eliminated in the limit as as i_0 approaches $\pm \sin^{-1}\sqrt{4/5}$. It is a remarkable aspect of this analysis that every term that appears to cause a solution to blow up is exactly canceled by a corresponding term that is "hidden" in the solution.

2. First Order Solution for y

By use of trigonometric identities, (6.23) may be rewritten as

$$\begin{aligned} y &= \theta - \omega + J\theta\left(\frac{5}{2}s^2 - 2\right) \\ &- Je^2\frac{(75s^6 - 260s^4 + 296s^2 - 112)}{96\left(\frac{5}{2}s^2 - 2\right)^2}\cos\left(2\omega - J\theta\left(\frac{5}{2}s^2 - 2\right)\right)\sin\left(J\theta\left(\frac{5}{2}s^2 - 2\right)\right) \\ &- \frac{e^2J^2\theta\cos(2\omega)}{30\left(\frac{5}{2}s^2 - 2\right)} + -J\theta_0\left(\frac{5}{2}s^2 - 2\right) + J^2\theta\bar{y}_2 + O(J^2, J^3\theta, + \ldots)\,. \end{aligned} \tag{6.34}$$

Equation (6.34) is the complete first order solution for y. If the initial inclination is such that $|\frac{5}{2}s^2 - 2| \leq 10^{-3}$, then (6.34) should be replaced by its

limit:

$$
\begin{aligned}
y &= \theta - \omega + \frac{e^2 J^3 \sin(2\omega)\theta^2}{15} + \frac{8eJ^2\cos(\omega-\theta_0)\theta}{15} + \frac{17e^2J^2\cos(2\omega)\theta}{60} \\
&+ \frac{e^2J^2\theta}{60} + \frac{2J^2\theta}{5} + O\left(J^2, J^3\theta, J^4\theta^3\left(\frac{5}{2}s^2-2\right)+\ldots\right).
\end{aligned}
$$

3. First Order Solution for i

As was done with the expression for y, i is rewritten as:

$$
\begin{aligned}
i &= i_0 + J\,c\,s\,\Bigg(\frac{2}{3}e[\cos(\theta_0-\omega)-\cos y] \\
&- \frac{e^2(15s^2-14)\sin\left[2\omega - J\theta\left(\frac{5}{2}s^2-2\right)\right]\sin\left[J\theta\left(\frac{5}{2}s^2-2\right)\right]}{24\left(\frac{5}{2}s^2-2\right)}\Bigg) \\
&+ O(J^2+J^3\theta+J^4\theta^2+\ldots). \qquad (6.35)
\end{aligned}
$$

Again, as in y, should i_0 be near the critical inclinations, $|\frac{5}{2}s^2-2| \leq 10^{-3}$, then (6.35) should be replaced by:

$$
\begin{aligned}
i &= i_0 + \frac{4Je}{15}[\cos(\theta_0-\omega)-\cos(\theta-\omega)] + \frac{J^2\theta e^2\sin 2\omega}{30} \\
&+ O\left(J^2, J^3\theta, J^3\theta^2\left(\frac{5}{2}\sin^2 i_0 - 2\right)+\ldots\right).
\end{aligned}
$$

4. First Order Solution for Ω

Rewriting (6.27) for Ω results in

$$
\begin{aligned}
\Omega &= \Omega_0 - J\theta c \\
&+ ce^2J\frac{(75s^4-120s^2+56)}{48\left(\frac{5}{2}s^2-2\right)^2} \\
&\quad \left[\cos\left(2\omega - J\theta\left(\frac{5}{2}s^2-2\right)\right)\sin\left(J\theta\left(\frac{5}{2}s^2-2\right)\right)\right] \\
&- \frac{4ceJ\sin y}{3} - \frac{ce^2J^2\cos(2\omega)\theta}{15s^2-12} + \frac{5ce^2J^2s^2\cos(2\omega)\theta}{8}
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{ce^2J^2\cos(2\omega)\theta}{12} + 5cJ^2K_2s^2\theta - \frac{5ce^2J^2s^2\theta}{24} + \frac{5cJ^2s^2\theta}{3} \\
& - \frac{ce^2J^2\theta}{6} + \frac{cJ^2\theta}{2} + K_4 + O(J^2, J^3\theta, J^4\theta^2 + \ldots) \qquad (6.36)
\end{aligned}
$$

where

$$
\begin{aligned}
K_2 &= \frac{2}{3}Jecs\cos(\theta_0 - \omega) \\
K_4 &= Jc\theta_0 + \frac{4}{3}Jce\sin(\theta_0 - \omega)\,.
\end{aligned}
$$

If $|\frac{5}{2}s^2 - 2| \leq 10^{-3}$, then

$$
\begin{aligned}
\Omega &= \Omega_0 - J\theta c - \frac{4ceJ\sin(\theta - \omega)}{3} + \frac{ce^2J^3\sin(2\omega)\theta^2}{6} + \frac{5ce^2J^2\cos(2\omega)\theta}{12} \\
&+ 4cJ^2K_2\theta - \frac{ce^2J^2\theta}{3} + \frac{11cJ^2\theta}{6} + K_4 \\
&+ O\left(J^2 + J^3\theta + J^4\theta^3\left(\frac{5}{2}s^2 - 2\right) + \ldots\right)
\end{aligned}
$$

where

$$
\begin{aligned}
K_2 &= \frac{4}{15}Je\cos(\theta_0 - \omega) \\
K_4 &= \sqrt{1/5}\,J\theta_0 + \frac{4}{3}\sqrt{1/5}\,Je\sin(\theta_0 - \omega)\,.
\end{aligned}
$$

5. First Order Solution for u

The complete first order solution for u is:

$$
\begin{aligned}
1 &+ e\cos y - \frac{e^2Js^2\cos(2y - 2\theta)}{24} + 2e^2JK_1s^2\cos(2y - 2\theta) \\
&+ \frac{e^2Js^2\cos(2y - 2\theta)}{8} - \frac{5eJs^2\cos(y + 2\theta)}{24} + \frac{11e^2Js^2\cos(2y)}{12} - eJK_7\cos(y - 2\theta) \\
&+ eJK_7\cos(y + 2\omega) - \frac{5e^2J\cos(2y)}{6} - \frac{e^2Js^2\cos(2\theta)}{4} - \frac{Js^2\cos(2\theta)}{6} \\
&- JK_9\sin(\omega - \theta_0) + JK_8\cos(\omega - \theta_0) - 2e^2JK_1s^2\cos(2\omega) - 2JK_2s^2 \\
&- \frac{17e^2Js^2}{12} - \frac{5Js^2}{2} + \frac{7e^2J}{6} + J + O(J^2) \qquad (6.37)
\end{aligned}
$$

where

$$K_1 = \frac{15s^2 - 14}{48\left(\frac{5}{2}s^2 - 2\right)}$$

$$K_2 = 2ecs\cos(\theta_0 - \omega)$$

$$\begin{aligned} y &= \theta - \omega + J\theta\left(\frac{5}{2}s^2 - 2\right) \\ &- Je^2\frac{(75s^6 - 260s^4 + 296s^2 - 112)}{48\left(\frac{5}{2}s^2 - 2\right)^2}\cos\left(2\omega - J\theta\left(\frac{5}{2}s^2 - 2\right)\right)\sin\left(J\theta\left(\frac{5}{2}s^2 - 2\right)\right) \\ &- e^2J^2\theta\cos(2\omega) + J\theta_0\left(\frac{5}{2}s^2 - 2\right) + J^2\theta\bar{y}_2 \quad \text{(see equation 6.22 for } \bar{y}_2\text{)} \end{aligned}$$

$$K_7 = \frac{-15e^2s^4 - (14e^2 - 2)s^2}{48\left(\frac{5}{2}s^2 - 2\right)}$$

K_8 -- Equation (6.32)

K_9 -- Equation (6.33)

Applying the condition $|\frac{5}{2}s^2 - 2| \leq 10^{-3}$, the expression for u is

$$\begin{aligned} u &= 1 - \frac{e^2J\cos(4\theta - 2\omega)}{30} + \frac{e^2J\cos(2\omega)}{10} - \frac{eJ\cos(3\theta - \omega)}{6} \\ &- \frac{e^2J\cos(2\theta - 2\omega)}{10} + JK_9\sin(\theta_0 - \omega) + JK_8\cos(\theta_0 - \omega) \\ &+ e\cos\left(\frac{e^2J^3\sin(2\omega)\theta^2}{15} + \frac{8eJ^2\cos(\omega - \theta_0)\theta}{15}\right. \\ &+ \left.\frac{17e^2J^2\cos(2\omega)\theta}{60} + \frac{e^2J^2\theta}{60} + \frac{2J^2\theta}{5} + \theta - \omega\right) \\ &- \frac{e(8e^2 - 8)J^2\theta\sin(\theta + \omega)}{120} - \frac{e^2J\cos(2\theta)}{5} - \frac{2J\cos(2\theta)}{15} \\ &- \frac{2e^2J^2\sin(2\omega)\theta}{15} - \frac{8JK_2}{5} + \frac{e^2J}{30} - J + O\left(J^2, J^3\theta, J^3\theta^2\left(\frac{5}{2}s^2 - 2\right) + \ldots\right). \end{aligned}$$

Where K_8 and K_9 are now:

$$K_8 = -\frac{e^2\cos(3\omega - \theta_0)}{20} - \frac{e^2\cos(3\omega - 3\theta_0)}{20} - \frac{e^2\cos(3\omega - 5\theta_0)}{20}$$

$$
\begin{aligned}
&- \frac{4e\cos(2\omega-2\theta_0)}{15} - \frac{3e\cos(2\omega-4\theta_0)}{10} - \frac{11e^2\cos(\omega+\theta_0)}{20} \\
&- \frac{\cos(\omega+\theta_0)}{5} - \frac{89e^2\cos(\omega-\theta_0)}{60} - \frac{\cos(\omega-\theta_0)}{5} \\
&- \frac{17e^2\cos(\omega-3\theta_0)}{60} - \frac{7\cos(\omega-3\theta_0)}{15} - \frac{2e\cos(2\omega)}{5} - e\cos(2\theta_0) \\
&- \frac{22e}{15} + O(J^2)
\end{aligned}
$$

$$
\begin{aligned}
K_9 = {} & \frac{e^2\sin(3\omega-\theta_0)}{20} + \frac{e^2\sin(3\omega-3\theta_0)}{20} + \frac{e^2\sin(3\omega-5\theta_0)}{20} \\
&+ \frac{4e\sin(2\omega-2\theta_0)}{15} + \frac{3e\sin(2\omega-4\theta_0)}{10} - \frac{7e^2\sin(\omega+\theta_0)}{20} \\
&+ \frac{\sin(\omega+\theta_0)}{5} + \frac{11e^2\sin(\omega-\theta_0)}{60} + \frac{\sin(\omega-\theta_0)}{5} + \frac{11e^2\sin(\omega-3\theta_0)}{60} \\
&+ \frac{7\sin(\omega-3\theta_0)}{15} + \frac{2e\sin(2\omega)}{5} - \frac{3e\sin(2\theta_0)}{5} + O(J^2)
\end{aligned}
$$

The results obtained thus far may be used to predict the orbit of a satellite for up to 1000 revolutions when a valid set of initial conditions are provided. The initial conditions will usually be given as the initial displacement vector $\mathbf{r}$ and the initial velocity $\mathbf{v}$, whereas the solution obtained is in terms of the orbital elements. It is important to show that the coordinates and velocity components can be easily recovered from the orbital elements. Indeed, Reference [4] contains the criticism that Brenner and Latta did not show how $\mathbf{r}$ and $\mathbf{v}$ would be obtained, and that "the derived elements are not such that the velocity components are readily obtainable." (Arsenault et al., Ref. 4: Vol. 2, p. 5).

Again referring back to Figure 5.4 and equations (5.1) - (5.3), the position of the satellite in the coordinate system may be derived from its direction cosines:

$$
\mathbf{r} = r\cos\delta\cos\phi\,\mathbf{i} + r\cos\delta\sin\phi\,\mathbf{j} + r\sin\delta\,\mathbf{k}
$$

where $\mathbf{r}$ is equation (6.30) and $\mathbf{i}, \mathbf{j}, \mathbf{k}$, are unit vectors.

Measuring from the line of nodes results in

$$\begin{aligned}
\mathbf{r} &= r\cos\delta\cos(\phi-\Omega)\mathbf{i}' + r\cos\delta\sin(\phi-\Omega)\mathbf{j}' + r\sin\delta\,\mathbf{k} \\
\text{where } \mathbf{i}' &= \cos\Omega\,\mathbf{i} + \sin\Omega\,\mathbf{j} \\
\mathbf{j}' &= -\sin\Omega\,\mathbf{i} + \cos\Omega\,\mathbf{j}.
\end{aligned}$$

Using the angle relationships from equation (5.1) - (5.3), the above expression is rewritten as

$$\begin{aligned}
\mathbf{r} &= r(\cos\theta\cos\Omega - \cos i\sin\theta\sin\Omega)\mathbf{i} \\
&+ r(\cos\theta\sin\Omega + \cos i\sin\theta\cos\Omega)\mathbf{j} + r\sin i\sin\theta\,\mathbf{k}\,. \qquad (6.38)
\end{aligned}$$

The velocity is found by differentiating (6.38) with respect to θ and noting

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{dt}\,.$$

The result is

$$\mathbf{v} = \left(\frac{dr_1}{d\theta}\mathbf{i} + \frac{dr_2}{d\theta}\mathbf{j} + \frac{dr_3}{d\theta}\mathbf{k}\right)\frac{d\theta}{dt}$$

where

$$\begin{aligned}
\frac{dr_1}{d\theta} &= -\cos i\sin\Omega\frac{dr}{d\theta}\sin\theta - \cos i\cos\Omega\frac{d\Omega}{d\theta}r\sin\theta \\
&+ \sin i\frac{di}{d\theta}\sin\Omega\, r\sin\theta - \cos\Omega\, r\sin\theta \\
&+ \cos\Omega\frac{dr}{d\theta}\cos\theta - \sin\Omega\frac{d\Omega}{d\theta}r\cos\theta \\
&- \cos i\sin\Omega\, r\cos\theta
\end{aligned}$$

$$\frac{dr_2}{d\theta} = r\left(-\cos i\sin\Omega\frac{d\Omega}{d\theta}\sin\theta - \sin\Omega\sin\theta\right.$$

$$
\begin{aligned}
- & \sin i \frac{di}{d\theta} \cos \Omega \sin \theta + \cos \Omega \frac{d\Omega}{d\theta} \cos \theta \\
+ & \cos i \cos \Omega \cos \theta \Big) \\
+ & \frac{dr}{d\theta} (\cos i \cos \Omega \sin \theta + \sin \Omega \cos \theta)
\end{aligned}
$$

$$
\frac{dr_3}{d\theta} = r \sin i \cos \theta + \sin i \sin \theta \frac{dr}{d\theta} + r \cos i \sin \theta \frac{di}{d\theta}.
$$

The requirement now is to find expressions for $\frac{di}{d\theta}$, $\frac{d\Omega}{d\theta}$, and $\frac{d\theta}{dt}$; the expression for $\frac{dr}{d\theta}$ is equation (6.31).

Differentiating the expression for i, equation (6.35), and the equation for Ω, equation (6.35), the following equations for $\frac{di}{d\theta}$ and $\frac{d\Omega}{d\theta}$ are obtained:

$$
\begin{aligned}
\frac{di}{d\theta} &= 2Jecs \sin y \\
\frac{d\Omega}{d\theta} &= -Jc - \frac{4}{3} Jce \sin y\,.
\end{aligned} \tag{6.39}
$$

The expression for $\frac{dt}{d\theta}$ is equation (6.29). By expanding this expression, the following equation for $\frac{d\theta}{dt}$ is obtained:

$$
\begin{aligned}
\frac{d\theta}{dt} &= \frac{\bar{h}}{r^2} \left[1 - J \left(-\frac{e^2 \cos(2y - 2\theta) - e^2 \cos(2\omega)}{15(5s^2 - 4)} + \frac{e^2 s^2 \cos(2y - 2\theta)}{8} \right. \right. \\
& - \frac{e^2 \cos(2y - 2\theta)}{60} - \frac{es^2 \cos(y + 2\theta)}{6} - \frac{es^2 \cos(y - 2\theta)}{2} \\
& + \frac{4es^2 \cos y}{3} - \frac{4e \cos y}{3} - \frac{s^2 \cos(2\theta)}{2} - \frac{2es^2 \cos(\omega - \theta_0)}{3} \\
& \left. \left. - \frac{e^2 s^2 \cos(2\omega)}{8} + \frac{e^2 \cos(2\omega)}{60} + \frac{s^2}{2} - 1 \right) \right].
\end{aligned}
$$

For $|\frac{5}{2}s^2 - 2| \leq 10^{-3}$

$$\frac{d\theta}{dt} = \frac{\bar{h}}{r^2}\left[1 + J\left(\frac{2e\cos(3\theta-\omega)}{15} + \frac{2e\cos(\theta+\omega)}{5} + \frac{4e\cos(\theta-\omega)}{15} + \frac{2\cos(2\theta)}{5}\right.\right.$$
$$\left.\left. + \frac{eJ\sin(2\omega)\theta}{15} + \frac{8e\cos(\omega-\theta_0)}{15} + \frac{3}{5}\right)\right].$$

D. VERIFICATION OF THE RESULTS

1. Comparison with Brenner and Latta

The comparison of solutions obtained by Brenner and Latta must be restricted to the J, Je, Je^2, and J^2 terms. Brenner and Latta limited their analysis in J to these terms and thus avoided the secular terms that arise in J^2e^2. No mention is made of the critical inclination in their work because $|\frac{5}{2}s^2 - 2|$ does not appear as a divisor in the terms they included. Similarly, this present analysis neglected all harmonics in the potential except J_2 while Brenner included some of the terms associated with the J_4 (or D_1 by their notation) harmonic.

Before comparing solutions, it should be noted that there is some difference between the two works in the notation used, this analysis preferring the more standard Astrodynamic notation. Brenner used the co-latitude angle (θ) as a spherical coordinate while the latitude (δ) was used here. In addition, Brenner used M as the independent variable where $M + \pi/2$ defines the angle from the ascending node to the satellite. The independent variable θ, measured from the ascending node to the satellite, was used here. Finally, Brenner defined the rotation of the line of nodes in an opposite sense to that done here. In summary,

$$\delta \rightarrow \pi/2 - \theta\,(\text{where}\,\theta\,\text{is co} - \text{latitude})$$

$$\theta \rightarrow M + \pi/2\,(\text{where}\,\theta\,\text{is the polar angle})$$

$$\Omega \rightarrow -\Omega$$

There will also be a difference in the constant terms. Brenner chose initial conditions that would make the analytical solutions as simple as possible. This analysis adopted the more general initial conditions that were outlined in Chapter 5.

Now restricting the comparison to the terms that are left, Brenner's solution for i and Ω are respectively:

$$
\begin{aligned}
i &= -\frac{2}{3}Jecs\cos y + \ldots \\
\Omega &= JMc - \frac{J^2Mc}{2} - \frac{5}{3}J^2Mcs^2 + \frac{4}{3}Jec\sin y + \frac{5J^2cs^2}{12}\sin(2m) + \ldots
\end{aligned}
$$

A check of equations (6.35) and (6.36) shows there is an exact match of terms of order J. The final term in Brenner's solution for Ω is a J^2 term and can be found in the solution for Ω_2, equation (B.4). Next, Brenner's abbreviated solution for u_1 is

$$\frac{5es^2}{24}\cos(y + 2M) + \frac{s^2}{6}\cos(2M) - \frac{5}{2}s^2 + 1 + \ldots$$

These same terms are duplicated in equation (6.17). Finally, Brenner's solution for y_1 is $\frac{5}{2}s^2 - 2$ which is exactly the expression obtained in this analysis.

2. Comparison with an Independent Analysis of Polar and Equatorial Orbits

Appendix A contains an independent analysis of the polar and equatorial orbits. Since for the polar case there is no variation of inclination, no rotation of the line of nodes, nor dependence of the motion on the longitude ϕ, new equations must be derived. The equations cannot be solved in terms of the angles i and Ω. Instead, the equations are solved in terms of the variable Δ^2 which is related to $\frac{dt}{d\theta}$.

For the equatorial case the inclination remains constant. The line of nodes is undefined since the satellite remains in the equatorial plane; however, the

angle Ω does change with time. Instead of using the polar angle θ to measure the angular displacement of the satellite, the angle ϕ, measured from the fixed axis γ, is used. A new dependent coordinate is defined as

$$Y = \phi - K + JY_1\phi + J^2Y_2\phi\,.$$

As Appendix A demonstrates, there is exact agreement between the special polar case and the general case for $i_0 = 90°$. In addition, when the appropriate change of variables is made in the equatorial case, it agrees exactly with the general case at $i_0 = 0^0$.

VII. CONCLUSIONS AND RECOMMENDATIONS

The results achieved in Chapter VI represent a unique solution to the problem of a satellite in orbit about an oblate planet. In fact, prior to this present work the problem had not been solved in this representation [Ref. 18, p. 340]. In contrast to the universal solution presented here, all other methods require a reformulation of the problem in alternate variables in order to achieve solutions at various singularities, the most prevalent being the critical inclination.

The results support the theory that at the critical inclination there are no discernable features in the perturbations of the elements that distinguish it from any other. This conclusion was reached by Lubowe [Ref. 38], is shared by Taff [Ref. 18] and has been corroborated by the physical data. However, all analytical solutions arrived at via canonic transformations predict perturbations in the vicinity of the critical to be 25 times greater than those away from it.

The perturbation method used in the analysis embodied the principles outlined in Chapter I. First, the resulting solutions are significantly more accurate than the two-body solution. For the appropriate orbits, the relative error of two-body solution is 1000 times greater than the solution obtained here. Second the solution was obtained in parameters closely related to the classical orbital elements and in Cartesian coordinates; no transformation to an alternate non-physical set of elements was required. Therefore the physical effects are easily distinguishable throughout the analysis. Finally, as has been noted the solution is valid for all orbital parameters.

While the perturbation method was similar to the Method of Strained Coordinates several novel techniques for dealing with secular terms and apparent singularities were introduced. The verification of these techniques was possible due to the extensive use of the symbolic computer program MACSYMA which allowed for the investigation of higher order formulas.

The analysis suggests several areas for further research. The areas include:

- The addition of the higher harmonics of the gravitational potential, e.g., J_3, J_4, etc., to the problem.
- The investigation into the feasibility of including non-conservative perturbing forces such as drag or solar radiation pressure.
- Numerically integrating the equations of motion subject to the initial conditions used here in order to check the analytic results.
- An in-depth analysis of the use of canonic transformations in satellite orbit prediction to include a comparison with the method used here, the purpose being to determine if the critical inclination singularity is an artifact of the transformation process.

APPENDIX A

THE POLAR AND EQUATORIAL ORBITS

A. THE POLAR ORBIT

Reference is made to the general equations of motion, equations (5.13) - (5.15). Now in the case of a polar orbit, the longitude ϕ is a constant. Therefore, the equations of motion are reduced to

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{-GM}{r^2} - 3J_2\frac{R^2}{r^4}\left(\frac{1}{2} - \frac{3}{2}\sin^2\theta\right) \qquad \text{(A.1)}$$

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -J_2\frac{GM\,R^2}{r^3}\sin(2\theta)\,. \qquad \text{(A.2)}$$

Let $J = \frac{3}{2}\left(J_2\frac{R^2}{p^2}\right)$ and $u = \bar{p}/r$

$$\text{where} \quad \bar{p} = \bar{h}^2/GM.$$

Define a function Δ such that

$$r^2\frac{d\theta}{dt} = \bar{h}\Delta$$

so that

$$\frac{d}{dt} = \frac{\bar{h}u^2\Delta}{\bar{p}^2}\frac{d}{d\theta}. \qquad \text{(A.3)}$$

Applying (A.3) to (A.2) the independent variable can be changed from t to θ. Equation (A.2) may be rewritten as

$$\frac{d\Delta^2}{d\theta} = -2\mathrm{Ju}\,\sin(2\theta). \qquad \text{(A.4)}$$

Similarly equation (A.1) becomes

$$\frac{d^2u}{d\theta} + u = \frac{1 + \mathrm{Ju}\left[\frac{du}{d\theta}\sin(2\theta) + u\frac{(3\cos 2\theta+1)}{2}\right]}{\Delta^2}\,. \qquad \text{(A.5)}$$

To solve (A.4) and (A.5) a solution of the form

$$\Delta^2 = 1 + JE_1 + J^2E_2 + \dots \tag{A.6}$$

$$u = 1 + e\cos y + Ju_1 + J^2u_2 + \dots \tag{A.7}$$

$$\text{where } y = \theta - \omega + J\theta y_1 + J^2\theta y_2 + \dots \tag{A.8}$$

is assumed.

Substitution of equations (A.6) and (A.7) into (A.4) and equating coefficients of order J results in:

$$\frac{dE_1}{d\theta} = -e\sin(y + 2\theta) + e\sin(y - 2\theta) - 2\sin(2\theta). \tag{A.9}$$

A solution of (A.9), ignoring terms of order J^2 or higher, is

$$\begin{aligned} E_1 &= \frac{e\cos(y+2\theta)}{3} + e\cos(y - 2\theta) + \cos(2\theta) \\ &+ K_1e^2\cos(2y - 2\theta). \end{aligned} \tag{A.10}$$

Substituting (A.10), (A.6), and (A.7) into (A.5) yields to order J

$$\begin{aligned} \frac{du_1}{d\theta} + u_1 &= 2e\cos y\, y_1 + \frac{5e^2(2y+2\theta)}{8} - e^2K_1\cos(2y - 2\theta) \\ &+ \frac{e^2\cos(2y-2\theta)}{8} + \frac{5e\cos(y+2\theta)}{3} - \frac{e^2\cos(2y)}{4} - e\cos(y) \\ &+ \frac{3e^2\cos(2\theta)}{4} + \frac{\cos(2\theta)}{2} - \frac{e^2}{4} - \frac{1}{2}. \end{aligned}$$

The cos y term would produce secular terms in the s .ion to u_1, therefore y_1 is chosen as $\frac{1}{2}$ to make the coefficient of cos y zero.

The resulting equation is

$$
\begin{aligned}
\frac{du_1}{d\theta} + u_1 &= \frac{5e^2\cos(2y+2\theta)}{8} - e^2K_1\cos(2y-2\theta) + \frac{e^2\cos(2y-2\theta)}{8} \\
&+ \frac{5e\,\cos(y+2\theta)}{3} - \frac{e^2\cos(2y)}{4} + \frac{3e^2\cos(2\theta)}{4} + \frac{\cos(2\theta)}{2} \\
&- \frac{e^2}{4} - \frac{1}{2}. \qquad \text{(A.11)}
\end{aligned}
$$

It is assumed that the solution to (A.11) has the following form:

$$
\begin{aligned}
u_1 &= a_o + a_1\cos(2y+2\theta) + a_2\cos(2y-2\theta) \\
&+ a_3\cos(y+2\theta) + a_5\cos(2\theta). \qquad \text{(A.12)}
\end{aligned}
$$

Equation (A.12) is substituted into (A.11). The unknown coefficients may then be solved for by equating the coefficients which have the same harmonics. The results are:

$$
\begin{aligned}
a_0 &= -\frac{e^2}{4} - \frac{1}{2} \\
a_1 &= -\frac{e^2}{24} \\
a_2 &= -\frac{(8K_1-1)e^2}{8} \\
a_3 &= -\frac{5e}{24} \\
a_4 &= \frac{e^2}{12} \\
a_5 &= -\frac{e^2}{4} - \frac{1}{6}.
\end{aligned}
$$

Substitution of these coefficients into (A.12) results in

$$
u_1 = -\frac{e^2\cos(2y+2\theta)}{24} - e^2K_1\cos(2y-2\theta) + \frac{e^2\cos(2y-2\theta)}{8}
$$

$$
\begin{aligned}
& - \frac{5e\cos(y+2\theta)}{24} + \frac{e^2\cos(2y)}{12} - \frac{e^2\cos(2\theta)}{4} - \frac{\cos(2\theta)}{6} \\
& - \frac{e^2}{4} - \frac{1}{2}.
\end{aligned}
$$

The following complementary solutions must be added to the particular solution for u to ensure that any secular terms in the second order solution can be eliminated

$$
-K_5\cos(3y-2\theta) - K_7\cos(y-2\theta).
$$

The first order solution for u now has the following form:

$$
\begin{aligned}
u \;=\; & 1 + e\cos y - J\,K_5\cos(3y-2\theta) - \frac{e^2 J\cos(2y+2\theta)}{24} \\
& - \frac{e^2 J(8K_1-1)\cos(2y-2\theta)}{8} - \frac{5eJ\cos(y+2\theta)}{24} - J\,K_7\cos(y-2\theta) \\
& + \frac{e^2 J\cos(2y)}{12} + J\,K_9\sin y + J\,K_8\cos y \\
& - \frac{(3e^2+2)J\cos(2\theta)}{12} - \frac{e^2 J}{4} - \frac{J}{2}.
\end{aligned} \tag{A.13}
$$

The expressions for E_1 and u_1 are substituted into (A.6) and (A 7) respectively. (A.6) and (A.7) are then substituted into (A.4). The result is

$$
\begin{aligned}
\frac{dE_2}{d\theta} \;=\; & -K_5\sin(3y-4\theta) + \frac{e^2\sin(2y+4\theta)}{24} - \frac{e^2\sin(2y+2\theta)}{12} \\
& + e^2K_1\sin(2y-2\theta) + \frac{e^2\sin(2y-2\theta)}{12} - e^2K_1\sin(2y-4\theta) \\
& + \frac{e^2\sin(2y-4\theta)}{8} + \frac{5e\sin(y+4\theta)}{24} - K_8\sin(y+2\theta) + \frac{e\sin(y+2\theta)}{6} \\
& + K_9\cos(y+2\theta) + K_8\sin(y-2\theta) + \frac{e\sin(y-2\theta)}{2} - K_9\cos(y-2\theta) \\
& - K_7\sin(y-4\theta) + K_5\sin(3y) + e^2K_1\sin(2y) - \frac{e^2\sin(2y)}{6} \\
& + K_7\sin y - \frac{5e\sin y}{24} + \frac{e^2\sin(4\theta Gx)}{4} + \frac{\sin(4\theta)}{6} \\
& + \frac{e^2\sin(2\theta)}{2} + \sin(2\theta).
\end{aligned}
$$

Integrating this expression for E_2 would yield a secular term in the harmonic $\cos(2y - 2\theta)$. Therefore the constant K_1 is chosen as $-\frac{1}{12}$ in order to eliminate the problem term. The equation is now

$$
\begin{aligned}
\frac{dE_2}{d\theta} &= -K_5 \sin(3y - 4\theta) + \frac{e^2 \sin(2y + 4\theta)}{24} - \frac{e^2 \sin(2y + 2\theta)}{12} \\
&+ \frac{5e^2 \sin(2y - 4\theta)}{24} + \frac{5e \sin(y + 4\theta)}{24} - K_8 \sin(y + 2\theta) \\
&+ \frac{e \sin(y + 2\theta)}{6} + K_9 \cos(y + 2\theta) + K_8 \sin(y - 2\theta) + \frac{e \sin(y - 2\theta)}{2} \\
&- K_9 \cos(y - 2\theta) - K_7 \sin(y - 4\theta) + K_5 \sin(3y) - \frac{e \sin(2y)}{4} \\
&+ K_7 \sin y - \frac{5e \sin y}{24} + \frac{e^2 \sin(4\theta)}{4} + \frac{\sin(4\theta)}{6} \\
&+ \frac{e^2 \sin(2\theta)}{2} + \sin(2\theta).
\end{aligned}
$$

Integrating the above expression yields

$$
\begin{aligned}
E_2 &= -K_5 \cos(3y - 4\theta) - \frac{e^2 \cos(2y + 4\theta)}{144} + \frac{e^2 \cos(2y + 2\theta)}{48} \\
&+ \frac{5e^2 \cos(2y - 4\theta)}{48} - \frac{e \cos(y + 4\theta)}{24} + \frac{K_9 \sin(y + 2\theta)}{3} \\
&+ \frac{K_8 \cos(y + 2\theta)}{3} - \frac{e \cos(y + 2\theta)}{18} + K_9 \sin(y - 2\theta) \\
&+ K_8 \cos(y - 2\theta) + \frac{e \cos(y - 2\theta)}{2} - \frac{K_7 \cos(y - 4x)}{3} - \frac{K_5 \cos(3y)}{3} \\
&+ \frac{e^2 \cos(2y)}{8} - K_7 \cos y + \frac{5e \cos y}{24} - \frac{e^2 \cos(4\theta)}{16} - \frac{\cos(4\theta)}{24} \\
&- \frac{e^2 \cos(2\theta)}{4} - \frac{\cos(2\theta)}{2}.
\end{aligned} \tag{A.14}
$$

Continuing the procedure results in the following second order expression

$$
\begin{aligned}
\frac{d^2 u_2}{d\theta^2} + u_2 &= 2e \cos y \, y_2 + \frac{eK_5 \cos(4y - 2\theta)}{2} - \frac{e^4 \cos(4y - 2\theta)}{96} \\
&- \frac{eK_5 \cos(4y - 4\theta)}{4} + \frac{5e^4 \cos(4y - 4\theta)}{576} - \frac{3e^3 \cos(3y + 4\theta)}{16}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{3e^3\cos(3y+2\theta)}{16} - 2K_5\cos(3y-2\theta) - \frac{e^3\cos(3y-2\theta)}{48} \\
&- \frac{35e^2\cos(2y+4\theta)}{32} + \frac{5eK_9\sin(2y+2\theta)}{4} + \frac{5eK_8\cos(2y+2\theta)}{4} \\
&+ \frac{5e^2\cos(2y+2\theta)}{12} + \frac{eK_9\sin(2y-2\theta)}{4} \\
&+ \frac{eK_8\cos(2y-2\theta)}{4} + \frac{eK_7\cos(2y-2\theta)}{2} + \frac{eK_5\cos(2y-2\theta)}{2} \\
&- \frac{e^4\cos(2y-2\theta)}{48} + \frac{7e^2\cos(2y-2\theta)}{24} \\
&- \frac{3eK_7\cos(2y-4\theta)}{4} - \frac{3eK_5\cos(2y-4\theta)}{4} + \frac{e^4\cos(2y-4\theta)}{32} \\
&+ \frac{e^2\cos(2y-4\theta)}{8} - \frac{7e^3\cos(y+4\theta)}{8} - \frac{7e\cos(y+4\theta)}{4} \\
&+ \frac{5K_9\sin(y+2\theta)}{3} + \frac{5K_8\cos(y+2\theta)}{3} \\
&- \frac{e^3\cos(y+2\theta)}{16} - \frac{13e\cos(y+2\theta)}{36} + 2K_7\cos(y-2\theta) \\
&+ \frac{e^3\cos(y-2\theta)}{16} - \frac{e\cos(y-2\theta)}{6} - \frac{5K_7\cos(y-4\theta)}{3} \\
&- \frac{55e^3\cos(y-4\theta)}{144} - \frac{5e\cos(y-4\theta)}{12} - \frac{5eK_5\cos(4y)}{4} \\
&+ \frac{5e^4\cos(4y)}{192} - \frac{5K_5\cos(3y)}{3} - \frac{13e^3\cos(3y)}{144} - \frac{eK_9\sin(2y)}{2} \\
&- \frac{eK_8\cos(2y)}{2} - \frac{3eK_7\cos(2y)}{4} - \frac{3eK_5\cos(2y)}{4} \\
&+ \frac{e^4\cos(2y)}{32} - \frac{23e^2\cos(2y)}{32} - \frac{25e^3\cos y}{48} \\
&- \frac{17e\cos y}{24} - \frac{5eK_7\cos(4\theta)}{4} + \frac{5e^4\cos(4\theta)}{192} - \frac{65e^2\cos(4\theta)}{32} \\
&- \frac{5\cos(4\theta)}{8} + \frac{3eK_8\cos(2\theta)}{2} + \frac{eK_7\cos(2\theta)}{2} \\
&- \frac{e^4\cos(2\theta)}{96} + \frac{37e^2\cos(2\theta)}{48} - \frac{\cos(2\theta)}{3} - \frac{eK_8}{2} \\
&- \frac{eK_7}{4} + \frac{5e^4}{576} - \frac{167e^2}{288} + \frac{1}{6}.
\end{aligned}
\tag{A.15}
$$

It is noted that three harmonics in (A.15) would produce secular terms in the solution to u_2. The harmonics are $\cos(3y - 2\theta)$, $\cos(y - 2\theta)$, and $\cos y$. These harmonics may be eliminated by choosing

$$
\begin{aligned}
K_5 &= -\frac{e^3}{96} \\
K_7 &= -\frac{(3e^3 - 8e)}{96} \\
y_2 &= \frac{25e^2 + 34}{96}.
\end{aligned}
$$

The first order solution for (A.4) is

$$
\begin{aligned}
\Delta^2 = 1 \quad &+ \quad J\Big(\cos(2\theta) + e\cos(y - 2\theta) \\
&+ \quad \frac{e\cos(y + 2\theta)}{3} - \frac{e^2\cos(2y - 2\theta)}{12}\Big). \qquad \text{(A.16)}
\end{aligned}
$$

u may now be rewritten as:

$$
\begin{aligned}
u \quad &= \quad 1 + e\cos y - \frac{e^2 J\cos(2y + 2\theta)}{24} + \frac{5e^2 J\cos(2y - 2\theta)}{24} - \frac{5eJ\cos(y + 2\theta)}{24} \\
&+ \quad \frac{e^3 J\cos(y - 2\theta)}{24} - \frac{eJ\cos(y - 2\theta)}{12} + \frac{e^2 J\cos(2y)}{12} - \frac{e^2 J\cos(2\theta)}{4} \\
&- \quad \frac{J\cos(2\theta)}{6} - J K_9 \sin(\omega - \theta_0) + J K_8 \cos(\omega - \theta_0) - \frac{e^2 J}{4} - \frac{J}{2} \qquad \text{(A.17)}
\end{aligned}
$$

$$
\text{where} \quad y = \theta - \omega + \frac{J\theta}{2} + \frac{Je^3\sin(2y - 2\theta)}{48} + J^2\theta\frac{(25e^2 + 34)}{96}.
$$

Equations (A.16) and (A.17) satisfy the equations of motion to first order, and the equations produce no secular terms to second order.

A comparison may now be made between the polar solution derived in this appendix and the general solutions for y and u, equations (6.22) and (6.37) respectively. To make the comparison easier, the relatively complex initial conditions

(equation 5.27) are discarded. Instead, following Brenner and Latta, the initial conditions are selected in order to make the expressions as simple as possible. In addition, constant terms like $\sin(2\omega)$ and $\cos(2\omega)$ are dropped. It will be remembered that these constants were added to the general solution to prevent singularities at the critical inclination. It is a simple exercise to add the $\cos(2\omega)$ term to the polar case, specifically to equation A.10, and then show that identical results are achieved with the general case at $i_o = 90°$. In fact, this has been done. However, to display these terms adds nothing to the comparison and only serves to make the analysis more tedious.

Checking equations (6.35) and (6.36), it is seen for the polar case, $\cos i_o = c = 0$, there results

$$i = i_o \qquad \text{and} \qquad \Omega = \Omega_o .$$

This agrees with orbital theory that for the polar case there is no variation of the inclination nor does the line of nodes rotate.

Setting $\sin i_o = s = 1$ and disregarding the appropriate constants, equations (6.37) and (6.22) are

$$\begin{aligned} 1 &+ e\cos y - \frac{e^2 J\cos(2y+2\theta)}{24} + \frac{5e^2 J\cos(2y-2\theta)}{24} - \frac{5eJ\cos(y-2\theta)}{24} \\ &+ \frac{e^3 J\cos(y-2\theta)}{24} - \frac{eJ\cos(y-2\theta)}{12} + \frac{e^2 J\cos(2y)}{12} - \frac{e^2 J\cos(2\theta)}{4} \\ &- \frac{J\cos(2\theta)}{6} - J K_9 \sin(\omega - \theta_0) + J K_8 \cos(\omega - \theta_0) - \frac{e^2 J}{4} - \frac{J}{2} \end{aligned}$$

and

$$y = \theta - \omega + \frac{J\theta}{2} + J\frac{e^3\sin(2y-2\theta)}{48} + J^2\theta\frac{(25e^2+34)}{96}.$$

These results agree exactly with the (A.17) and (A.18).

B. THE EQUATORIAL ORBIT

Reference is again made to the general equations of motion, (5.13) - (5.15). An orbit that remains precisely in the equatorial plane is possible if only even harmonics of the potential are considered. Now $\delta = 0°$, $\frac{\partial V}{\partial \delta} = 0$, $\frac{\partial V}{\partial \phi} = 0$, and the equations reduce to

$$\frac{d^2 r}{dt^2} - r\frac{d\phi}{dt} = -\frac{\partial V}{\partial r} \tag{A.18}$$

$$\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right) = 0. \tag{A.19}$$

As in the general case and the polar case the independent variable is changed by use of the integral obtained from (A.19)

$$\begin{aligned} r^2\frac{d\phi}{dt} &= \text{constant} = \bar{h} \\ \frac{d}{dt} &= \frac{u^2}{p^2}\bar{h}\frac{d}{d\phi}. \end{aligned} \tag{A.20}$$

Substitution of (A.20) into (A.19), results in

$$\frac{d^2 u}{d\phi^2} + u = 1 + J\,u^2 \tag{A.21}$$

$$\begin{aligned} \text{where:}\quad J &= \frac{3}{2}\left(J_2\frac{R^2}{p^2}\right) \\ u &= \bar{p}/r. \end{aligned}$$

To solve (A.21), assume a solution of the form

$$u = 1 + e\cos Y + Ju_1 + J^2u_2 + \ldots \tag{A.22}$$

$$\text{where}\quad Y = \phi - \omega + J\phi Y_1 + J^2\phi Y_2.$$

Substituting (A.22) into (A.21) and equating coefficients of order J results in

$$\frac{d^2 u_1}{d\phi^2} + u_1 = 1 + 2J\,Y_1 e\cos Y + 2Je\cos Y + e^2\cos^2 Y. \tag{A.23}$$

The presence of the term cos Y will cause secular terms in the solution for u_1, therefore choose $Y_1 = -1$. Equation (A.23) is now

$$\frac{d^2u_1}{d\phi^2} + u_1 = 1 + \frac{e^2}{2} + \frac{e^2\cos^2}{2}Y. \tag{A.24}$$

Assume a particular solution to the above equation

$$u_1 = \alpha_o + \alpha_1 \cos 2Y. \tag{A.25}$$

Substitution of (A.25) into (A.24) results in the following solution for u_1

$$u_1 = 1 + \frac{e^2}{2} - \frac{1}{6}e^2\cos(2Y)\,. \tag{A.26}$$

The approximation for u is:

$$u = 1 + e\cos Y + J\left(1 + \frac{e^2}{2} - \frac{1}{6}e^2\cos(2Y)\right). \tag{A.27}$$

The solution must be checked to ensure it causes no secular terms to second order. Substituting (A.26) and (A.22) into (A.21) results in

$$\begin{aligned}\frac{d^2u_2}{d\phi^2} + u_2 &= -\frac{e^3\cos(2Y)}{6} + e^2\cos(2Y) + 2e\cos Y\,Y_2\\ &+ \frac{5e^3\cos Y}{6} + 3e\cos Y + e^2 + 2.\end{aligned}$$

Note that the terms with cos Y will cause secular terms in u_2. Therefore, Y_2 is chosen as:

$$-\frac{5e^2}{12} - \frac{3}{2}.$$

The first order solution for u is equation (A.27)

$$\text{where}\quad Y = \phi - \omega - J\phi - J^2\phi\left(\frac{5e^2}{12} + \frac{3}{2}\right). \tag{A.28}$$

Equation (A.27) is the correct solution in terms of the variable ϕ. To compare the solution found here to the general solution for the case $\phi = 0°$ will require that (A.27) and (A.28) be modified such that they are in terms of θ.

Referring back to Figure (5.1), it is seen that ϕ is measured from the fixed axis Υ such that

$$\phi = \pi/2 + \theta + \Omega. \tag{A.29}$$

Now, from equation (6.27), the first order solution for Ω with $i_o = 0°$ ($\sin i_o = s = 0$ and $\cos i_o = c = 1$) is

$$\Omega = \Omega_o - J\theta + \frac{7Je^2}{48}\sin(2y - 2\theta) - \frac{4}{3}Je\sin y - \frac{J^2e^2\theta}{6} + \frac{J^2\theta}{2}. \tag{A.30}$$

(Note: As in the polar case, the constant terms $\cos(2\omega)$ and $\sin(2\omega)$ have been dropped in the general case for comparison purposes. As explained previously, eliminating these terms does not affect the validity of the comparison.)

Substitution of (A.30) and (A.29) into (A.28) results in

$$Y = \theta - \omega - 2J\theta + \frac{7Je^2}{48}\sin(2y - 2\theta) - \frac{7}{12}J^2e^2\theta - \frac{4}{3}Je\sin y. \tag{A.31}$$

Now, from (6.22), the solution for y with $i_o = 0°$ is

$$y = \theta - \omega - 2J\theta - \frac{7}{12}J^2e^2\theta + \frac{7}{48}Je^2\sin(2y - 2x). \tag{A.32}$$

Using equation (A.32), equation (A.31) may be written as

$$Y = y - \frac{4}{3}Je\sin y.$$

And therefore by use of the Taylor series expansion

$$\cos(Y) = \cos y + \frac{4}{3}Je\sin^2 y.$$

Equation (A.27) may now be written in terms of $y(\theta)$:

$$u = 1 + e\cos y + J\left(1 + \frac{7}{6}e^2 - \frac{5}{6}e^2\cos 2y\right). \tag{A.33}$$

Reference now is made to equation (6.37), the general solution for u. By letting $\sin i_o = s = 0$ in this equation, it is easily seen that the general expression becomes (A.33).

APPENDIX B

2nd ORDER ANALYSIS

Substitution of (6.2) - (6.5), (6.10), (6.16), and (6.17) into (5.23) results in

$$
\begin{aligned}
&- 2\sin i\frac{d\Omega}{d\theta}\sin\theta - \frac{d^2 i_2}{d\theta^2}\sin\theta + \sin i\frac{d^2\Omega_2}{d\theta^2}\cos\theta - 2\frac{di_2}{d\theta}\cos\theta \\
&= ce^3K_1s^3\sin(3y-\theta) - cK_5s\sin(3y-\theta) + \frac{ce^3K_1 s\sin(3y-\theta)}{2} \\
&- \frac{5ce^3K_1s^3\sin(3y-3\theta)}{3} + cK_5s\sin(3y-3\theta) + \frac{ce^3K_1s\sin(3y-3\theta)}{6} \\
&- \frac{7ce^2s^3\sin(2y+3\theta)}{24} + \frac{143ce^2s^3\sin(2y+\theta)}{72} - \frac{29ce^2s\sin(2y+\theta)}{18} \\
&+ 5ce^2K_3s^3\sin(2y-\theta) + 10ce^2K_1s^3\sin(2y-\theta) - \frac{15ce^2s^3\sin(2y-\theta)}{8} \\
&- 4ce^2K_3s\sin(2y-\theta) - 4ce^2K_1s\sin(2y-\theta) + \frac{7ce^2s\sin(2y-\theta)}{6} \\
&- 5ce^2K_3s^3\sin(2y-3\theta) + \frac{5ce^2s^3\sin(2y-3\theta)}{8} + 4ce^2K_3s\sin(2y-3\theta) \\
&- 4ce^2K_1s\sin(2y-3\theta) - \frac{35ces^3\sin(y+3\theta)}{24} + ceK_7s\sin(y+\theta+2\omega) \\
&- 2ce^3K_1s^3\cos(2\omega)\sin(y+\theta) - ce^3K_1s\cos(2\omega)\sin(y+\theta) \\
&- 2ceK_2s^3\sin(y+\theta) + \frac{5ces^2\sin(y+\theta)}{8} + 2cK_8s\sin(y+\theta) \\
&- ceK_2s\sin(y+\theta) - \frac{4ce\,s\sin(y+\theta)}{3} - 2cK_9s\cos(y+\theta) \\
&- ceK_7s\sin(y-\theta+2\omega) + \frac{10ce^3K_1s^3\cos(2\omega)\sin(y-\theta)}{3} \\
&- \frac{ce^3K_1s\cos(2\omega)\sin(y-\theta)}{3} + \frac{10ceK_2s^3\sin(y-\theta)}{3} + \frac{5ce^3K_1s^3\sin(y-\theta)}{3} \\
&- \frac{41ce\,s^3\sin(y-\theta)}{12} - 2cK_8s\sin(y-\theta) - ceK_7s\sin(y-\theta) \\
&- \frac{ceK_2s\sin(y-\theta)}{3} - \frac{ce^3K_1s\sin(y-\theta)}{6} + \frac{8ces\sin(y-\theta)}{3}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 2cK_9 s\cos(y-\theta) - ce^3K_1s^3\sin(y-3\theta) + \frac{9ces^3\sin(y-3\theta)}{4} \\
&+ \quad ceK_7 s\sin(y-3\theta) - \frac{ce^3K_1 s\sin(y-3\theta)}{2} - \frac{5ce^2s^3\sin(3\theta)}{4} - \frac{5cs^3\sin(3\theta)}{3} \\
&- \quad 10ce^2K_1s^3\cos(2\omega)\sin\theta - 10cK_2s^3\sin\theta + \frac{5ce^2s^3\sin\theta}{12} \\
&- \quad \frac{10cs^3\sin\theta}{3} + \frac{ce^2\sin\theta}{3} - cs\sin\theta. \qquad \text{(B.1)}
\end{aligned}
$$

Choose $K_1 = \frac{15s^2-14}{24(5s^2-4)}$ and $K_3 = \frac{75s^4-120s^2+56}{24(5s^2-4)^2}$ to eliminate the secular terms $\sin(2y-\theta)$ and $\sin(2y-3\theta)$ in equation (B.1), so now

$$
\begin{aligned}
&- \quad 2\sin i\frac{d\Omega_2}{d\theta}\sin\theta - \frac{d^2i_2}{d\theta^2}\sin\theta + \sin i\frac{d^2\Omega_2}{d\theta^2}\cos\theta - 2\frac{di_2}{d\theta}\cos\theta \\
&= \quad -\frac{39ce^3s\sin(3y-\theta)}{1800s^2-1440} + \frac{ce^3s^3\sin(3y-\theta)}{8} - cK_5s\sin(3y-\theta) \\
&+ \quad \frac{11ce^3s\sin(3y-\theta)}{240} + \frac{35ce^3s\sin(3y-3\theta)}{1800s^2-1440} - \frac{5ce^3s^3\sin(3y-3\theta)}{24} \\
&+ \quad cK_5s\sin(3y-3\theta) + \frac{7ce^3s\sin(3y-3\theta)}{144} - \frac{7ce^2s^3\sin(2y+3\theta)}{24} \\
&+ \quad \frac{143ce^2s^3\sin(2y+\theta)}{72} - \frac{29ce^2s\sin(2y+\theta)}{18} - \frac{35ces^3\sin(y+3\theta)}{24} \\
&+ \quad ceK_7s\sin(y+\theta+2\omega) + \frac{78ce^3s\cos(2\omega)\sin(y+\theta)}{1800s^2-1440} \\
&- \quad \frac{ce^3s^3\cos(2\omega)\sin(y+\theta)}{4} - \frac{11ce^3s\cos(2\omega)\sin(y+\theta)}{120} \\
&- \quad 2ceK_2s^3\sin(y+\theta) + \frac{5ces^3\sin(y+\theta)}{8} + 2cK_8s\sin(y+\theta) \\
&- \quad ceK_2s\sin(y+\theta) - \frac{4ces\sin(y+\theta)}{3} - 2cK_9s\cos(y+\theta) \\
&- \quad ceK_7s\sin(y-\theta+2\omega) - \frac{70ce^3s\cos(2\omega)\sin(y-\theta)}{1800s^2-1440} \\
&+ \quad \frac{5ce^3s^3\cos(2\omega)\sin(y-\theta)}{12} - \frac{7ce^3s\cos(2\omega)\sin(y-\theta)}{72} \\
&- \quad \frac{35ce^3s\sin(y-\theta)}{1800s^2-1440} + \frac{10ceK_2s^3\sin(y-\theta)}{3} + \frac{5ce^3s^3\sin(y-\theta)}{24}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{41ces^3\sin(y-\theta)}{12} - 2cK_8 s\sin(y-\theta) - ceK_7 s\sin(y-\theta) \\
&- \frac{ceK_2 s\sin(y-\theta)}{3} - \frac{7ce^3 s\sin(y-\theta)}{144} + \frac{8ces\sin(y-\theta)}{3} + 2cK_8 s\cos(y-\theta) \\
&+ \frac{39ce^3 s\sin(y-3\theta)}{1800s^2-1440} - \frac{ce^3 s^3\sin(y-3\theta)}{8} + \frac{9ces^3\sin(y-3\theta)}{4} \\
&+ ceK_7 s\sin(y-3\theta) - \frac{11ce^3 s\sin(y-3\theta)}{240} - \frac{5ce^2 s^3\sin(3\theta)}{4} - \frac{5cs^3\sin(3\theta)}{3} \\
&+ \frac{240ce^2 s\cos(2\omega)\sin\theta}{1800s^2-1440} - \frac{5ce^2 s^3\cos(2\omega)\sin\theta}{4} + \frac{ce^2 s\cos(2\omega)\sin\theta}{6} \\
&- 10cK_2 s^3\sin\theta + \frac{5ce^2 s^3\sin\theta}{12} - \frac{10cs^3\sin\theta}{3} + \frac{ce^2 s\sin\theta}{3} - cs\sin\theta\,. \qquad \text{(B.2)}
\end{aligned}
$$

Solve for i_2 and Ω_2 using the technique described in Chapter 6, Section 5. The second order expressions for i_2 and Ω_2 are respectively

$$
\begin{aligned}
i_2 = {} & \frac{11ce^3 s\cos(3y-2\theta)}{900s^2-720} - \frac{ce^3 s^3\cos(3y-2\theta)}{6} + \frac{23ce^3 s\cos(3y-2\theta)}{360} \\
&- \frac{2cK_5 s\cos(3y-2\theta)}{3} - \frac{25ce^2 s^3\cos(2y+2\theta)}{96} + \frac{29ce^2 s\cos(2y+2\theta)}{144} \\
&+ \frac{11ce^3 s\cos(y-2\theta)}{900s^2-720} - \frac{2cK_9 s\sin y}{3} - \frac{22ce^3 s\cos(2\omega)\cos y}{900s^2-720} \\
&+ \frac{ce^3 s^3\cos(2\omega)\cos y}{3} - \frac{23ce^3 s\cos(2\omega)\cos y}{180} + \frac{8ceK_2 s^3\cos y}{3} \\
&+ \frac{ce^3 s^3\cos y}{6} - \frac{85ces^3\cos y}{36} - \frac{2cK_8 s\cos y}{3} - \frac{2cK_7 s\cos y}{3} \\
&- \frac{2ceK_2 s\cos y}{3} - \frac{23ce^3 s\cos y}{360} + \frac{20ces\cos y}{9} \qquad \text{(B.3)}
\end{aligned}
$$

$$
\begin{aligned}
\Omega_2 = {} & \frac{2ce^3\sin(3y-2\theta)}{75s^2-60} - \frac{ce^3 s^2\sin(3y-2\theta)}{4} + \frac{ce^3\sin(3y-2\theta)}{30} \\
&+ \frac{4cK_5\sin(3y-2\theta)}{3} - \frac{17ce^2 s^2\sin(2y+2\theta)}{72} + \frac{29ce^2\sin(2y+2\theta)}{144} \\
&+ \frac{7ces^2\sin(y+2\theta)}{36} - \frac{2ce^3\sin(y-2\theta)}{75s^2-60} + \frac{ce^3 s^2\sin(y-2\theta)}{12} \\
&- \frac{3ces^2\sin(y-2\theta)}{2} - \frac{2cK_7\sin(y-2\theta)}{3} + \frac{11ce^3\sin(y-2\theta)}{360}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{4ce^3\cos(2\omega)\sin y}{75s^2-60} + \frac{ce^3s^2\cos(2\omega)\sin y}{2} - \frac{ce^3\cos(2\omega)\sin y}{15} \\
&+ 4ceK_2s^2\sin y + \frac{ce^3s^2\sin y}{6} - \frac{107ces^2\sin y}{36} - \frac{4cK_8\sin y}{3} \\
&- \frac{2cK_7\sin y}{3} - \frac{23ce^3\sin y}{360} + \frac{28ce\sin y}{9} + \frac{4cK_9\cos y}{3} \\
&+ \frac{5ce^2s^2\sin(2\theta)}{16} + \frac{5cs^2\sin(2\theta)}{12} - \frac{5ce^2\cos(2\omega)\theta}{75s^2-60} + \frac{5ce^2s^2\cos(2\omega)\theta}{8} \\
&- \frac{ce^2\cos(2\omega)\theta}{12} + 5cK_2s^2\theta - \frac{5ce^2s^2\theta}{24} + \frac{5cs^2\theta}{3} - \frac{ce^2\theta}{6} + \frac{c\theta}{2}. \qquad \text{(B.4)}
\end{aligned}
$$

Substitution of (6.2) - (6.5), (6.10), (6.16), (6.17), (B.3), and (B.4) into equation (5.24) results in

$$
\begin{aligned}
\frac{d^2u_2}{d\theta^2} &+ u_2 = 2e\cos y\, y_2 - \frac{120e^4\cos(4y-2\theta)}{36000s^2-28800} - \frac{5e^4s^4\cos(4y-2\theta)}{16} \\
&+ \frac{7eK_5s^2\cos(4y-2\theta)}{2} + \frac{31e^4s^2\cos(4y-2\theta)}{96} - 3eK_5\cos(4y-2\theta) \\
&- \frac{e^4\cos(4y-2\theta)}{240} + \frac{e^4\cos(4y-4\theta)}{15000s^4-24000s^2+9600} + \frac{9e^4\cos(4y-4\theta)}{36000s^2-28800} \\
&+ \frac{29e^4s^4\cos(4y-4\theta)}{192} - \frac{3eK_5s^2\cos(4y-4\theta)}{4} - \frac{923e^4s^2\cos(4y-4\theta)}{5760} \\
&+ \frac{eK_5\cos(4y-4\theta)}{2} + \frac{89e^4\cos(4y-4\theta)}{7200} - \frac{3e^3s^4\cos(3y+4\theta)}{16} \\
&- \frac{e^3s^4\cos(3y+2\theta)}{48} + \frac{21e^3s^2\cos(3y+2\theta)}{8} - \frac{29e^3\cos(3y+2\theta)}{12} \\
&- \frac{480e^3\cos(3y-2\theta)}{36000s^2-28800} + \frac{5e^3s^4\cos(3y-2\theta)}{16} - 10K_5s^2\cos(3y-2\theta) \\
&- \frac{5e^3s^2\cos(3y-2\theta)}{6} + 8K_5\cos(3y-2\theta) + \frac{17e^3\cos(3y-2\theta)}{30} \\
&- \frac{35e^2s^4\cos(2y+4\theta)}{32} + \frac{3eK_7s^2\cos(2y+2\theta+2\omega)}{4} \\
&+ \frac{3eK_5s^2\cos(2y+2\theta-2\omega)}{4} + \frac{360e^4\cos(2\omega)\cos(2y+2\theta)}{36000s^2-28800} \\
&- \frac{5e^4s^4\cos(2\omega)\cos(2y+2\theta)}{16} + \frac{19e^4s^2\cos(2\omega)\cos(2y+2\theta)}{96}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{e^4\cos(2\omega)\cos(2y+2\theta)}{80} - \frac{5e^2K_2s^4\cos(2y+2\theta)}{2} + \frac{95e^2s^4\cos(2y+2\theta)}{12} \\
&+ \frac{5e^2K_2s^2\cos(2y+2\theta)}{4} - \frac{65e^2s^2\cos(2y+2\theta)}{8} + \frac{3eK_7s^2\cos(2y-2\theta+2\omega)}{4} \\
&+ \frac{3eK_5s^2\cos(2y-2\theta-2\omega)}{4} - \frac{4e^4\cos(2\omega)\cos(2y-2\theta)}{15000s^4-24000s^2+9600} \\
&+ \frac{172e^4\cos(2\omega)\cos(2y-2\theta)}{36000s^2-28800} - \frac{e^4s^4\cos(2\omega)\cos(2y-2\theta)}{8} \\
&+ \frac{7e^4s^2\cos(2\omega)\cos(2y-2\theta)}{80} + \frac{23e^4\cos(2\omega)\cos(2y-2\theta)}{3600} \\
&+ \frac{576e^2K_2\cos(2y-2\theta)}{36000s^2-28800} - \frac{164e^4\cos(2y-2\theta)}{36000s^2-28800} + \frac{2400e^2\cos(2y-2\theta)}{36000s^2-28800} \\
&- e^2K_2s^4\cos(2y-2\theta) - \frac{e^4s^4\cos(2y-2\theta)}{2} + \frac{257e^2s^4\cos(2y-2\theta)}{72} \\
&+ \frac{5eK_7s^2\cos(2y-2\theta)}{2} + \frac{13eK_5s^2\cos(2y-2\theta)}{6} + \frac{17e^2K_2s^2\cos(2y-2\theta)}{30} \\
&+ \frac{29e^4s^2\cos(2y-2\theta)}{60} - \frac{245e^2s^2\cos(2y-2\theta)}{72} - 2eK_7\cos(2y-2\theta) \\
&- \frac{5eK_5\cos(2y-2\theta)}{3} + \frac{e^2K_2\cos(2y-2\theta)}{50} + \frac{67e^4\cos(2y-2\theta)}{3600} \\
&+ \frac{e^2\cos(2y-2\theta)}{12} - \frac{6e^4\cos(2y-4\theta)}{36000s^2-28800} - \frac{528e^2\cos(2y-4\theta)}{36000s^2-28800} \\
&+ \frac{e^4s^4\cos(2y-4\theta)}{2} + \frac{5e^2s^4\cos(2y-4\theta)}{16} - \frac{3eK_7s^2\cos(2y-4\theta)}{4} \\
&- \frac{9eK_5s^2\cos(2y-4\theta)}{4} - \frac{121e^4s^2\cos(2y-4\theta)}{240} - \frac{23e^2s^2\cos(2y-4\theta)}{240} \\
&+ \frac{3eK_5\cos(2y-4\theta)}{2} + \frac{29e^4\cos(2y-4\theta)}{800} - \frac{11e^2\cos(2y-4\theta)}{600} \\
&- \frac{3eK_7s^2\cos(2y+2\omega)}{2} + eK_7\cos(2y+2\omega) - \frac{3eK_5s^2\cos(2y-2\omega)}{2} \\
&+ eK_5\cos(2y-2\omega) - \frac{7e^3s^4\cos(y+4\theta)}{8} - \frac{7es^4\cos(y+4\theta)}{4} \\
&+ \frac{7K_7s^2\cos(y+2\theta+2\omega)}{3} + \frac{7K_5s^2\cos(y+2\theta-2\omega)}{3}
\end{aligned}
$$

$$+ \frac{2240e^3\cos(2\omega)\cos(y+2\theta)}{36000s^2-28800} - \frac{5e^3s^4\cos(2\omega)\cos(y+2\theta)}{4}$$

$$+ \frac{7e^3s^2\cos(2\omega)\cos(y+2\theta)}{12} + \frac{7e^3\cos(2\omega)\cos(y+2\theta)}{90}$$

$$- 10eK_2s^4\cos(y+2\theta) + \frac{661e^3s^4\cos(y+2\theta)}{144} + \frac{47es^4\cos(y+2\theta)}{36}$$

$$+ \frac{10eK_2s^2\cos(y+2\theta)}{3} - \frac{131e^3s^2\cos(y+2\theta)}{24} - 5es^2\cos(y+2\theta)$$

$$+ \frac{29e^3\cos(y+2\theta)}{36} + 2K_7s^2\cos(y-2\theta+2\omega) + 2K_5s^2\cos(y-2\theta-2\omega)$$

$$- \frac{480e^3\cos(y-2\theta)}{36000s^2-28800} + \frac{25e^3s^4\cos(y-2\theta)}{16} + 10K_7s^2\cos(y-2\theta)$$

$$- 2e^3s^2\cos(y-2\theta) - \frac{es^2\cos(y-2\theta)}{6} - 8K_7\cos(y-2\theta) + \frac{17e^3\cos(y-2\theta)}{30}$$

$$- \frac{1120e^3\cos(y-4\theta)}{36000s^2-28800} + \frac{5e^3s^4\cos(y-4\theta)}{48} - \frac{53s^4\cos(y-4\theta)}{12}$$

$$- \frac{5K_7s^2\cos(y-4\theta)}{3} - \frac{7e^3s^2\cos(y-4\theta)}{24} - \frac{7e^3\cos(y-4\theta)}{180} - \frac{570e^4\cos(4y)}{36000s^2-28800}$$

$$- \frac{5e^4s^4\cos(4y)}{32} + \frac{5eK_5s^2\cos(4y)}{4} + \frac{19e^4s^2\cos(4y)}{192} - \frac{5eK_5\cos(4y)}{2}$$

$$+ \frac{13e^4\cos(4y)}{80} - \frac{1120e^3\cos(3y)}{36000s^2-28800} - \frac{269e^3s^4\cos(3y)}{48} - \frac{5K_5s^2\cos(3y)}{3}$$

$$+ \frac{679e^3s^2\cos(3y)}{72} - \frac{677e^3\cos(3y)}{180} + \frac{816e^4\cos(2\omega)\cos(2y)}{36000s^2-28800}$$

$$+ \frac{11e^4s^4\cos(2\omega)\cos(2y)}{8} - \frac{359e^4s^2\cos(2\omega)\cos(2y)}{240}$$

$$+ \frac{17e^4\cos(2\omega)\cos(2y)}{600} - \frac{6e^4\cos(2y)}{36000s^2-28800} - \frac{528e^2\cos(2y)}{36000s^2-28800}$$

$$+ 11e^2K_2s^4\cos(2y) + \frac{e^4s^4\cos(2y)}{2} - \frac{695e^2s^4\cos(2y)}{96} - \frac{7eK_7s^2\cos(2y)}{4}$$

$$- \frac{5eK_5s^2\cos(2y)}{4} - \frac{21e^2K_2s^2\cos(2y)}{2} - \frac{121e^4s^2\cos(2y)}{240}$$

$$+ \frac{3007e^2s^2\cos(2y)}{240} + eK_7\cos(2y) + \frac{eK_5\cos(2y)}{2} + \frac{29e^4\cos(2y)}{800}$$

$$
\begin{aligned}
&- \frac{1137e^2\cos(2y)}{200} + \frac{960e^3\cos(2\omega)\cos y}{36000s^2-28800} + \frac{15e^3s^4\cos(2\omega)\cos y}{4} \\
&- \frac{15e^3s^2\cos(2\omega)\cos y}{4} + \frac{e^3\cos(2\omega)\cos y}{30} + 30eK_2s^4\cos y \\
&- \frac{15e^3s^4\cos y}{16} + \frac{275es^4\cos y}{24} - 26eK_2s^2\cos y - \frac{3e^3s^2\cos y}{4} \\
&- \frac{61es^2\cos y}{6} + \frac{7e^3\cos y}{6} + \frac{5eK_7s^2\cos(2\theta+2\omega)}{4} + \frac{eK_5s^2\cos(2\theta+2\omega)}{4} \\
&+ \frac{eK_7s^2\cos(2\theta-2\omega)}{4} + \frac{5eK_5s^2\cos(2\theta-2\omega)}{4} - \frac{180e^4\cos(4\theta)}{36000s^2-28800} \\
&+ \frac{5e^4s^4\cos(4\theta)}{32} - \frac{65e^2s^4\cos(4\theta)}{32} - \frac{5s^4\cos(4\theta)}{8} - \frac{5eK_7s^2\cos(4\theta)}{4} \\
&- \frac{19e^4s^2\cos(4\theta)}{192} - \frac{e^4\cos(4\theta)}{160} + \frac{432e^4\cos(2\omega)\cos(2\theta)}{36000s^2-28800} \\
&+ \frac{1056e^2\cos(2\omega)\cos(2\theta)}{36000s^2-28800} - \frac{3e^4s^4\cos(2\omega)\cos(2\theta)}{8} \\
&- \frac{e^2s^4\cos(2\omega)\cos(2\theta)}{2} + \frac{19e^4s^2\cos(2\omega)\cos(2\theta)}{80} \\
&+ \frac{23e^2s^2\cos(2\omega)\cos(2\theta)}{120} + \frac{3e^4\cos(2\omega)\cos(2\theta)}{200} + \frac{11e^2\cos(2\omega)\cos(2\theta)}{300} \\
&- \frac{276e^4\cos(2\theta)}{36000s^2-28800} - 3e^2K_2s^4\cos(2\theta) - 4K_2s^4\cos(2\theta) - \frac{7e^4s^4\cos(2\theta)}{16} \\
&+ \frac{127e^2s^4\cos(2\theta)}{16} - \frac{7s^4\cos(2\theta)}{6} + \frac{9eK_7s^2\cos(2\theta)}{2} + \frac{3e^2K_2s^2\cos(2\theta)}{2} \\
&+ K_2s^2\cos(2\theta) + \frac{193e^4s^2\cos(2\theta)}{480} - \frac{95e^2s^2\cos(2\theta)}{12} - \frac{2s^2\cos(2\theta)}{3} \\
&- 4eK_7\cos(2\theta) + \frac{19e^4\cos(2\theta)}{300} + \frac{2e^4\cos(2\omega)^2}{15000s^4-24000s^2+9600} - \frac{50e^4\cos(2\omega)^2}{36000s^2-28800} \\
&+ \frac{e^4s^4\cos(2\omega)^2}{32} - \frac{23e^4s^2\cos(2\omega)^2}{960} - \frac{7e^4\cos(2\omega)^2}{3600} - \frac{576e^2K_2\cos(2\omega)}{36000s^2-28800}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{304e^4\cos(2\omega)}{36000s^2-28800} - \frac{2400e^2\cos(2\omega)}{36000s^2-28800} + \frac{e^2K_2s^4\cos(2\omega)}{2} + \frac{17e^4s^4\cos(2\omega)}{24} \\
&+ \frac{5e^2s^4\cos(2\omega)}{2} - \frac{3eK_7s^2\cos(2\omega)}{2} - \frac{3eK_5s^2\cos(2\omega)}{2}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{19e^2K_2s^2\cos(2\omega)}{60} - \frac{533e^4s^2\cos(2\omega)}{720} - \frac{41e^2s^2\cos(2\omega)}{24} + eK_7\cos(2\omega) \\
&+ eK_5\cos(2\omega) - \frac{e^2K_2\cos(2\omega)}{50} + \frac{19e^4\cos(2\omega)}{1800} - \frac{e^2\cos(2\omega)}{12} \\
&+ \frac{e^4}{15000s^4 - 24000s^2 + 9600} - \frac{17e^4}{36000s^2 - 28800} + 2K_2^2s^4 + \frac{17e^2K_2s^4}{3} + 20K_2s^4 \\
&+ \frac{25e^4s^4}{192} - \frac{437e^2s^4}{288} + \frac{97s^4}{8} - \frac{7eK_7s^2}{12} - K_2^2s^2 - \frac{31e^2K_2s^2}{6} - 11K_2s^2 - \frac{847e^4s^2}{5760} \\
&+ \frac{139e^2s^2}{72} - 10s^2 + \frac{eK_7}{3} + \frac{17e^4}{720} - \frac{11e^2}{9} + 2\,. \qquad \text{(B.5)}
\end{aligned}
$$

APPENDIX C

THE CRITICAL INCLINATION

This appendix contains a discussion of the critical inclination to include its mathematical and physical basis and the methods used by various authors to deal with the problem.

In Chapter 6 it was found that the divisor $\left(\frac{5\sin^2 i_0 - 2}{2}\right)$ often appeared in second order terms. A brief explanation was given for the appearance of this divisor, and it was shown that the apparent singularity may be dealt with in this analysis in a straight-forward fashion. However, in other methods the problem divisor has caused enormous complications. This divisor or "critical inclination" problem appears to be universal. Hughes' investigations [Ref. 32] reveal that every analytical theory on orbit perturbation analysis contains the critical inclination as an inherent characteristic. As stated in Chapter 2, the divisor causes an irremovable singularity in Kozai's method. Its presence has prompted many authors to devise very elaborate techniques to get around the problem with the result that many otherwise elegant methods are somewhat diminished.

The mathematical reason for the appearance of the problem term is as follows. It is recalled that one of the secular effects of the Earth's equatorial bulge is to cause the line of apsides to rotate. A first order approximation for this rotation is given by the rate of change of the argument of perigee which from Allan [Ref. 33] is

$$\frac{d\omega}{dt} = \frac{3J_2 n}{4(1-e^2)^2}\frac{R^2}{a^2}(5\cos^2 i - 1)$$

It is readily seen that if $i = 63°26'$, the perigee position is stationary and the line of apsides does not rotate. When the usual Poisson method of successive

approximations in terms of a small parameter is adopted for perturbation analysis, the above equation appears in the denominator of higher order terms.

Coffey [Ref. 34] gives an excellent background discussion on the history of the critical inclination problem. He states that A. A. Orlov (1957) was perhaps the first to notice the unusual situation at the critical inclination. Orlov found that the conventional Lindstedt-Poincaré algorithm failed at this critical inclination and at zero inclination. Nevertheless, Orlov ignored these exceptional cases and developed his theories without addressing the problem of singularities. Krause (1952), Roberson (1957), and Herget and Musen (1958) had all overlooked this difficulty in their assessment of the long term effects of the J_2 term on the Keplerian ellipse before Brouwer (1958) finally pointed out the problem to the latter authors. Brouwer hoped that his use of von Ziepel's method of eliminating variables through canonic transformations would allow him to dispose of the terms that lead to singularities. However, as was pointed out in Chapter 2, his solution contained singularities at, not only the critical inclination, but also zero inclination or eccentricity.

Finding no method to fit the critical inclination problem into current theories, astronomers have devised alternative theories to deal with orbital inclinations near 63°.4. The critical inclination, they reasoned, was a small divisor problem and therefore could be handled in the standard way introduced by Bohlen [Ref. 35]. That is, in the vicinity of the critical inclination, the solution may be expressed in terms of the square root of a small parameter, i.e., $\sqrt{J}$ instead of J. [Ref. 36]. The resulting solutions are totally different in character to the non-singular case [Ref. 30]. For example, instead of precessing secularly the argument of perigee will tend to oscillate about a central position. Taff notes the lack of mathematical rigor in this method of handling the critical inclination by stating that expansions

in $\sqrt{J}$ and theories about the libration of perigee are a result of misapplication of perturbation theories [Ref. 18, p 340].

Usually when singularities arise in celestial mechanics there is a physical reason for their occurrence. For example, singularities in the solutions for the motion of asteroids and natural satellites are caused by the fact that their orbital periods are nearly commensurate with that of the disturbing body. Under the same circumstances, an artificial satellite may encounter resonant perturbations in its orbital elements caused by tesseral harmonics. However, such a physical reason does not seem to suggest itself for the critical inclination problem. Message [Ref. 37] implied that there may be a resonance between the satellite's mean motion relative to perigee and its mean motion relative to the ascending node; however, most skeptics have not been convinced [Ref. 32]. If there were resonance one would expect to be able to experimentally measure deviations caused by resonances, but satellite guidance engineers have reported no adverse effects at the critical inclination.

Another check on the physical effects of resonance should be a numerical check of the equations of motion at the critical inclination; however, this too has been inconclusive. Lubowe [Ref. 38] carried out a study in which he integrated a satellite's equations of motion for periods up to 24 hours with various initial conditions at inclinations near and away from the critical. He found no discernible features in the perturbations which distinguished the critical from any other. However, in a separate analysis, Hughes came to the opposite conclusion. Hughes developed the Hamiltonian in terms of the Hill variables and showed that the critical inclination remained an inherent part of the solution throughout a multitude of various canonic transformations. He then integrated the equations of motion numerically and was able to show some resonant effects in the perigee

height for satellites at the critical inclination. He then reasoned from this analysis that the critical inclination is unique, that it represents actual physical resonance and that it is not merely a by-product of the method of solution or the type of variable used in the analysis.

This disagreement characterizes the lingering debate over the question of whether the critical inclination is an artifact of the analysis process or a physical reality. Most textbooks choose to mention the problem; however, few voice an opinion. Roy [Ref. 20] devotes only two sentences to the subject, merely noting that some perturbation analyses break down there. Hagihara [Ref. 39], mentions that the critical inclination is an essential feature of the perturbation process but notes that heated discussions continue concerning its physical reality. Other authors, namely Herrick [Ref. 40], Kovalevsky [Ref. 41], and Geyling [Ref. 1] mention the problem and the standard procedure for dealing with it, but they make no mention of physical effects. Taff [Ref. 18, p 340] takes a firmer position by stating that it is not a prediction of second-order perturbation theory that there are infinitely large or infinitely rapid changes in the orbital elements, and that the correct resolution of these "unphysical" predictions is not to rely on bad mathematics.

This present analysis lends support to Taff's assertion. Although the critical inclination problem was manifested in the perturbation process, it was shown that it does not cause singularities, nor are there any unusual physical effects predicted by this theory. In fact, it was a remarkable aspect of this analysis that in the limit as the inclination approached the critical, all potential singularities were canceled. This is not true of the analyses which use the technique of Delauney/von Ziepel canonic transformations. A remarkable aspect of the canonic transformation method is that the non-integrability of the equations of motion

is not obvious when the system is expressed in coordinates or elements, but after the canonical transformation it is quite clear [Ref. 42]. The critical inclination is an intrinsic singularity of the method [Ref. 32].

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